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ABSTRACT

This seventh unit in the SMSG junior high mathematics series is the teacher's commentary for Unit 5. A time allotment for each of the chapters in Unit 5 is suggested. Then, for each of the chapters in Unit 5, the objectives for that chapter are specified, the mathematics is discussed, some teaching suggestions are provided, the answers to exercises are listed, and sample test questions for that chapter are suggested. (DT)

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SMSG

SE 021 618

School Mathematics Study Group

Mathematics for Junior High School, Volume 2

Unit 7

Mathematics for Junior High School, Volume 2

Teacher's Commentary, Part I

Prepared under the supervision of the
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PREFACE

Key ideas of junior high school mathematics emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school.

Fourteen experimental units for use in the seventh and eighth grades were written in the summer of 1958 and tried out by approximately 100 teachers in 12 centers in various parts of the country in the school year 1958-59. On the basis of teacher evaluations these units were revised during the summer of 1959 and, with a number of new units, were made a part of sample textbooks for grade 7 and a book of experimental units for grade 8. In the school year 1959-60, these seventh and eighth grade books were used by about 175 teachers in many parts of the country, and then further revised in the summer of 1960. Again during the year 1960-61, this text for grade 8 was used by nearly 200 classes in all parts of the country, and then this edition was prepared in the summer of 1961.

Mathematics is fascinating to many persons because of its opportunities for creation and discovery as well as for its utility. It is continuously and rapidly growing under the prodding of both intellectual curiosity and practical applications. Even junior high school students may formulate mathematical questions and conjectures which they can test and perhaps settle; they can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. Recognition of these important factors has played a considerable part in selection of content and method in this text.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that this text may greatly assist all teachers who use it to achieve this highly desirable goal.

The preliminary edition of this volume was prepared at a writing session held at the University of Michigan during the summer of 1959. Revisions were prepared at Stanford University in the summer of 1960, taking into account the classroom experience with the preliminary edition during the academic year 1959-60. This edition was prepared at Yale University in the summer of 1961, again taking into account the classroom experience with the Stanford edition during the academic year 1960-61.

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NOTE TO TEACHERS

Based on the teaching experience of over 100 junior high school teachers in all parts of the country and the estimates of authors of the revisions (including junior high school teachers), it is recommended that teaching time for Part 1 be as follows:

Chapter	Approximate number of days
1	15
2	15
3	15
4	13-15
5	6
6	15
Total	79-81

Teachers are urged to try not to exceed these approximate time allotments so that pupils will not miss the chapters at the end of the course. Some classes will be able to finish certain chapters in less than the estimated time.

Throughout the text, problems, topics, and sections which were designed for the better students are indicated by an asterisk (*). Items starred in this manner should be used or omitted as a means of adjusting the approximate time schedule.

Chapter 1

RATIONAL NUMBERS AND COORDINATES

General Remarks

This chapter is intended to provide an informal introduction to negative rational numbers. The aim is to present the negative numbers as real, useful quantities which have already been used by the students for some time. The examples and illustrations have been selected to advance this point of view, and the teacher can develop enthusiastic interest by emphasizing applications of this sort.

Although the approach is informal, the important ideas about negative numbers have been included. With a few exceptions, the discussion of formal properties of the negative rationals has been placed in the exercises. It is expected that some classes (especially those that have done well on the Volume I SMSG material) will enjoy working on these formal aspects of the number system. The problems suggested in the exercises provide a good basis for such a treatment. On the other hand, classes without previous experience with SMSG texts may find these problems difficult at first. This will be especially true when the chapter is being taught at the beginning of the school year.

Our major aim here is to achieve a real understanding and familiarity with negative rationals without insisting upon complete mastery of formal techniques and details. The student should be encouraged to take time to figure things out by reference to the number line or to any of the other devices which have been introduced to make the negatives plausible to him.

The sections on coordinates and graphs capitalize on the use of the number line in introducing negative numbers. The introduction of graphs of linear equations and inequalities provides practice in the use of negative numbers.

The teacher should think of this chapter in relation to both Chapter 2 on equations and to Chapter 6 on real numbers. Much can be done in the class discussion to anticipate the problems of these later chapters and to provide a background for their study.

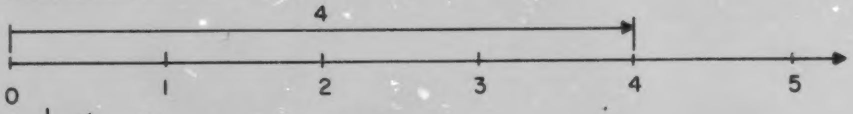
It is expected that 15 days class time should be ample for most classes to cover this chapter. Students who have just completed the first volume of SMSG material will probably be able to complete the chapter in a shorter time.

1-1. The Number Line

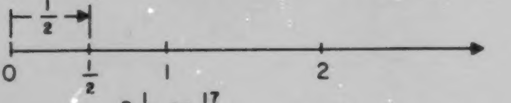
This discussion of the number line is a rather complete one, intended to interest the student by relating the number to its associated point and to the corresponding directed line segment. Students who have studied SMSG Mathematics for Junior High School, Volume I, will probably be able to cover this section very rapidly. It provides a good review of material in Chapters 3, 6 and 8 of Volume I. Teachers who have not taught the SMSG seventh grade program will probably wish to read these three chapters (3, 6, 8) before teaching this material.

Answers to Exercises 1-1

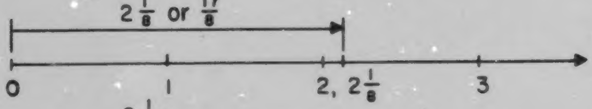
1. (a)



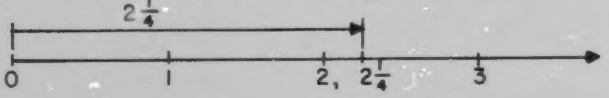
(b)



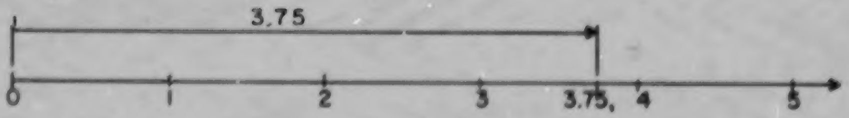
(c)



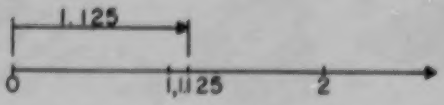
(d)



(e)


3.75
0
1
2
3
3.75,
4
5

(f)


1.125
0
1,1125
2

2. Drawings are omitted in order to conserve space. In each case the sketch is similar to the one given in the student text.

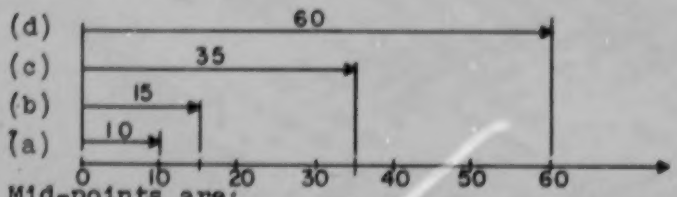
The teacher should emphasize that for the problems of this section, a scale drawing is unnecessary. A rough sketch is sufficient.

3. From right to left the points are:

(a) 1, $\frac{9}{8}$, $\frac{3}{2}$ (c) 3, $\frac{13}{4}$, $\frac{7}{2}$

(b) $\frac{3}{4}$, $\frac{13}{16}$, $\frac{8}{8}$

4.


(d)
(c)
(b)
(a)
60
35
15
10
0
10
20
30
40
50
60

5. Mid-points are:

(a) 1 (c) 2

(b) $\frac{3}{8}$ (d) 4.25

6. Commutative property.

7. Associative property.

8. $3 \cdot 2 = 2 + 2 + 2$. That is, multiplication is repeated addition.

9. Commutative property.

1-2. Negative Rational Numbers

The negative numbers have been introduced with reference to the number line to provide geometric reality and an intuitive feeling for them. Discussions of their use in situations describing "opposite" characteristics should contribute also to a feeling of familiarity and usefulness. The idea of "opposite" is one deserving special emphasis.

The opposite characteristics should be stressed. The ordering of the rational numbers on the number line has been introduced and is worth continued discussion. It is valuable to develop a natural acceptance of the fact that "is less than" means the same as "precedes on the number line." To say that one number "is greater than" another means simply that the one "follows" the other in the number line.

Note that the idea of magnitude, which suggests the absolute value, of a number has been avoided. This idea is, of course, implicit in the length of the arrow associated with a rational number. It seems less troublesome, however, to avoid specific mention of absolute value in this informal treatment.

The student should be encouraged to provide sketches for the numbers with which he deals. To save his time and to focus attention on true understanding, he should be encouraged to draw a rough sketch which need not be carefully scaled. It is only important that relative lengths and relative positions should be faithfully represented. An intelligent sketch will often help the student's understanding more than a carefully drawn scale used in a routine way. In this section, and in the one which follows, the student is getting a valuable introduction to operations with directed line segments or vectors. Although not mentioned specifically, the notion of directed distances is being developed in this incidental way.

In this chapter a number like "negative three" is written as $^{-}3$, or $(^{-}3)$. The use of parentheses is more helpful in some contexts than in others. For example in

$$4 + (^{-}3) \text{ or } 4 - (^{-}3)$$

[page 7]

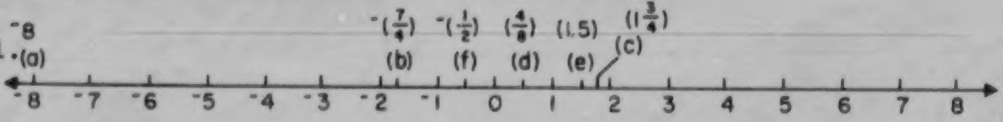
the use of parentheses in this way should make the meaning clearer. With negative rational numbers which are not integers the authors recommend that the notation $^{-}(\frac{3}{4})$ be used exclusively until the study of division. At that time students will learn that

$$^{-}(\frac{3}{4}) = \frac{^{-}3}{4} = \frac{3}{^{-}4}.$$

The authors have not been consistent in the use of parentheses with negative numerals. When a lowered hyphen is used, parentheses must be used consistently. When a raised hyphen is used, as in this book, parentheses may be omitted when there is no ambiguity.

Answers to Exercises 1-2

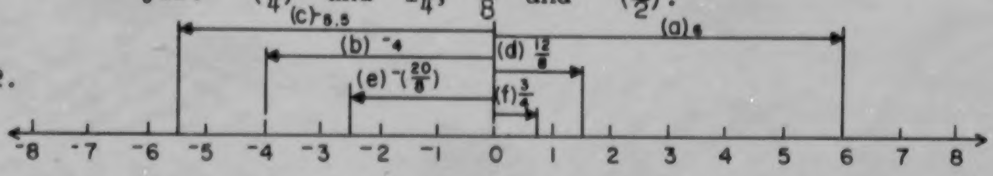
1.



⁻8
(a)
⁻(7/4)
(b)
⁻(1/2)
(f)
(4/8)
(d)
(1.5)
(e)
(1 3/4)
(c)
⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6 7 8

yes. $^{-}(\frac{7}{4})$ and $1\frac{3}{4}$; $\frac{4}{8}$ and $^{-}(\frac{1}{2})$.

2.



(c) ⁻5.5
(a) 6
(b) ⁻4
(d) 12/8
(e) ⁻(20/8)
(f) 3/4
⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6 7 8

3. $^{-}6$, $^{-}4$, $^{-}(\frac{7}{4})$, $^{-}(\frac{3}{8})$, $\frac{1}{4}$, $\frac{5}{8}$, $\frac{3}{4}$

largest is $\frac{3}{4}$

smallest is $^{-}6$

4. (a) $^{+}2000$ (c) $^{-}15$

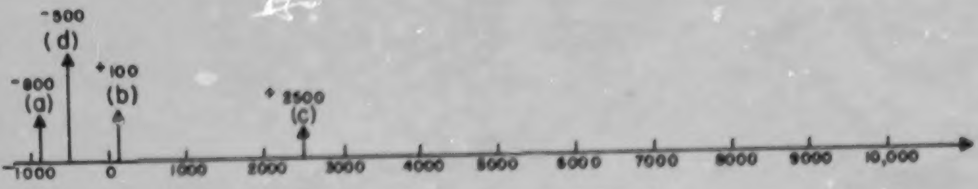
$^{-}6000$ $^{+}10$

(b) $^{+}100$ (d) $^{+}2$

$^{-}50$ $^{-}4$

5. $^{-}2$, $^{-}1$, 0, $^{+}1$, $^{+}2$, $^{+}3$

6.



1-3. Addition of Rational Numbers

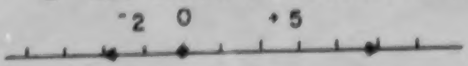
The treatment of addition continues the emphasis on the number line and the use of arrows to represent positive and negative rationals. Here, especially, the student should be encouraged to sketch in a rough but essentially accurate fashion. A proper grounding at this stage should give the student something to fall back on in later periods of perplexity.

Our initial discussion of the use of arrows in the positive number line did not include a discussion of subtraction by means of arrows. Nor do we at this stage discuss the equivalence between the addition of $(^-3)$ and the subtraction of $(^+3)$. Unless the issue is unavoidably introduced into class discussion, it would seem preferable to avoid discussion of these matters at this stage.

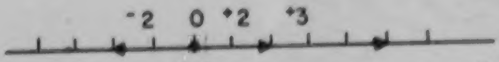
The idea of the additive inverse is an important one. It will be used in Chapter 2.

It is important that teachers understand that the drawing on Page 20 is not intended to diagram the sum $5 + (^-2)$ in the same way as other sums were diagrammed earlier in this section. In this figure the additive inverse relationship of 2 and $^-2$ is being emphasized. The net effect of adding $^+5$ and $^-2$ may be thought of as follows.

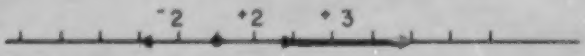
The $^+5$ and the $^-2$ are shown as arrows on the number line.



The number "5" may be thought of as $(^+2 + ^+3)$.



Since $^+2$ is the additive inverse of $^-2$, or the "true opposite" of $^-2$, we can picture the $^+2$ as an arrow equal in length to $^-2$, but in the opposite direction.



Since $(^-2) + (^+2) = 0$, the arrow representing the $^+2$ is, in effect, removed from the arrow representing the $^+5$. The remaining arrow, representing $^+3$, is the sum.

Answers to Exercises 1-3a

As before, drawings are omitted in order to conserve space. In each case, the sketch is similar to one in the text.

1.	(a)	4	(d)	$^-5$
	(b)	3	(e)	$^-5$
	(c)	3	(f)	$^-8$
2.	(a)	0	(e)	$\frac{1}{2}$
	(b)	4	(f)	$\frac{2}{3}$
	(c)	$^-6$	(g)	12
	(d)	$^-1$	(h)	0
3.	(a)	19	(d)	$^-30$
	(b)	$^-12$	(e)	$^-6$
	(c)	$^-5$	(f)	3
4.	(a)	$^-5$	(d)	2
	(b)	$^-9$	(e)	$^-8$
	(c)	$^-11$	(f)	$^-6$

5. (a)

January	$^{+}5000$	April	$^{+}1000$
February	$^{+}2000$	May	$^{-}4000$
March	$^{-}6000$	June	$^{-}3000$

(b) $^{-}5000$ or \$5000 loss

(c) $^{+}1000$ or \$1000 profit

(d) $^{-}12,000$ or \$12,000 loss

6. (a) $^{+}4$ rate of boy upstream
$^{-}2$ rate of current downstream

(b) $^{+}2$ resultant rate of boy upstream

7. (a) $^{+}17$, $^{-}6$, $^{+}11$, $^{-}3$

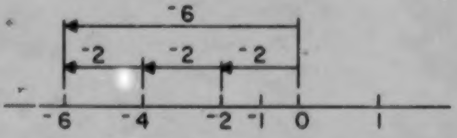
(b) $20 + 17 = 37$
$37 + (^{-}6) = 31$
$31 + 11 = 42$
$42 + (^{-}3) = 39$
ball is on the 39 yard line

(c) $17 + (^{-}6) + 11 + (^{-}3) = 19$. The net gain is 19 yards.

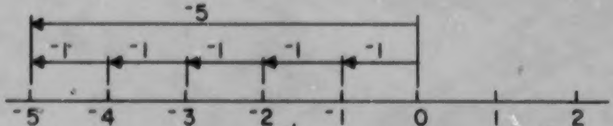
8. (a) $13,000 + 5000 + (^{-}3000) = 15,000$

(b) 15,000 ft.

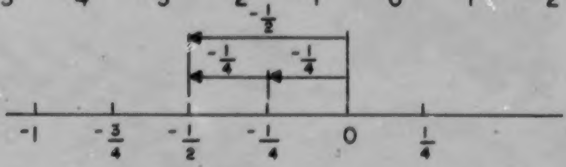
9. (a) $3(^{-}2) = {}^{-}6$


-6
-2
-2
-2
-6
-4
-2
-1
0
1

(b) $5(^{-}1) = {}^{-}5$


-5
-1
-1
-1
-1
-1
-5
-4
-3
-2
-1
0
1
2

(c) $2 \cdot {}^{-}(\frac{1}{4}) = {}^{-}(\frac{1}{2})$


-1/2
-1/4
-1/4
-1
-3/4
-1/2
-1/4
0
1/4

Answers to Class Exercises 1-3a

1. $^-7$, 9, $^-11$, 12, 6, $^-15$, 20, 0, $\frac{2}{3}$, $^-(\frac{4}{9})$, $\frac{7}{8}$, $^-(\frac{30}{31})$

2. The pairs in (c) and in (d) are additive inverses.

Answers to Class Exercises 1-3b

1. Sketches are similar to the one in the text.

2. (a) 5 (c) $^-1$
   (b) 2 (d) $^-4$

Answers to Exercises 1-3b

1.	(a)	3	(d)	5, 18
	(b)	40	(e)	$^-16$, $^-20$
	(c)	$^-12$		
2.	(a)	30	(d)	$^-11$
	(b)	$^-4$	(e)	204
	(c)	28	(f)	$^-76$

3. (a)	2	positive	(h)	$^-(\frac{1}{10})$	negative
(b)	$^-20$	negative	(i)	$(\frac{1}{14})$	positive
(c)	$^-2$	negative	(j)	$^-0.1188$	negative
(d)	12	positive	(k)	0.0004	positive
(e)	2	positive	(ℓ)	$^-8.9988$	negative
(f)	$\frac{1}{8}$	positive	(m)	0.225	positive
(g)	$^-(\frac{3}{16})$	negative	(n)	$^-(\frac{1}{48})$	negative

4. False. Since any positive number is greater than any negative number, the sign of the greater number is always positive; but the sum may be negative or zero.

5. ... they are additive inverses.

1-4. Coordinates

The use of a number line in introducing positive and negative rational numbers establishes, essentially, a system of coordinates on the line. The principal new idea here is the use of the term, coordinate, and the notation (2) to designate a coordinate of the point to which the number 2 has been assigned on the number line. Note that we have actually defined "a coordinate" rather than "the coordinate" although we have not made an issue of maintaining this distinction. A point on the line will have a different coordinate if the position of the origin is changed or if the unit of length is changed. Indeed, coordinate systems could be defined in which the scale, or method of assigning numbers to points, is not uniform. One might, for example, think of the scale on a slide rule as being a kind of number scale.

Credit for the invention of a coordinate system in a plane is often given to the French mathematician and philosopher, René Descartes. The method used by Descartes first appeared in a mathematical treatise published in 1637. More recently historians of science have pointed out that the essence of the idea appeared in the earlier work of several mathematicians. In any case, coordinate geometry is of considerably more recent origin than the Euclidean geometry of the traditional high school course.

Just as the number line provides for the association of a number with a point on a number line, a coordinate system in the plane provides for the association of a pair of numbers with a point in the plane. In the pair the numbers are ordered. There is a one-to-one correspondence between all rational number pairs and some of the points in the plane. In a coordinate system as described in this chapter, there are many points with which no ordered number pair of rational numbers is associated. An example is $(\sqrt{2}, \sqrt{2})$.

As with a coordinate of a point on a line we will speak of a pair of coordinates which is an ordered number pair, associated with a point in the plane.

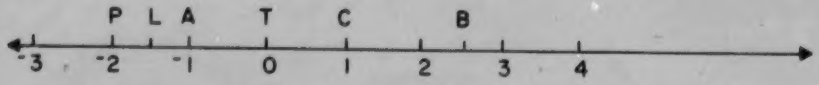
In this section students should learn (a) to locate the point when coordinates of the point are given, and (b) to give an ordered number pair associated with a point, when the point is given.

It is recommended that students be encouraged to use care in drawing the axes and labeling the scales on the axes. It is suggested that the numbering of the scale be below the X-axis and to the left of the Y-axis. If the axes are drawn with arrows, it will help the students to keep in mind that the axes are lines, not line segments.

Always label the X-axis and the Y-axis as shown in the diagrams in the student's text, at least until students have had quite a bit of experience in plotting points and graphs of equations.

Answers to Exercises 1-4a

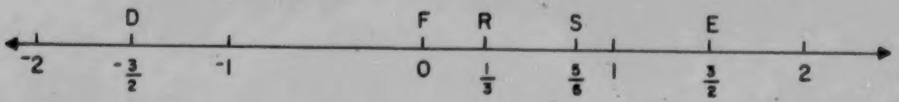
1.



2. (a) $1\frac{1}{2}$"　　(c) 4"

   (b) $4\frac{1}{2}$"　　(d) 1"

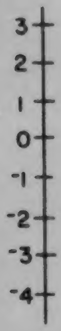
3.



4. (a) $\frac{1}{3}$ mile

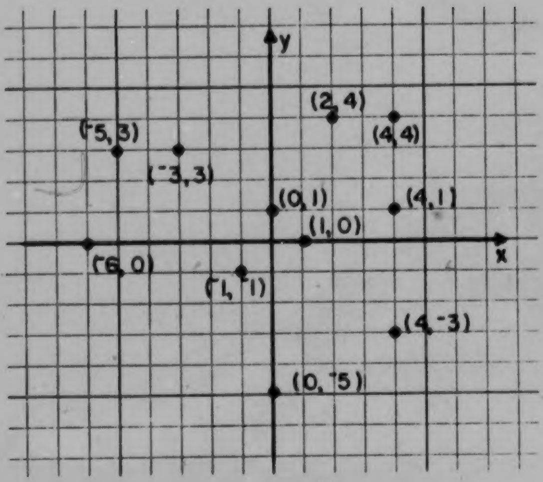
   (b) 3 miles

5.

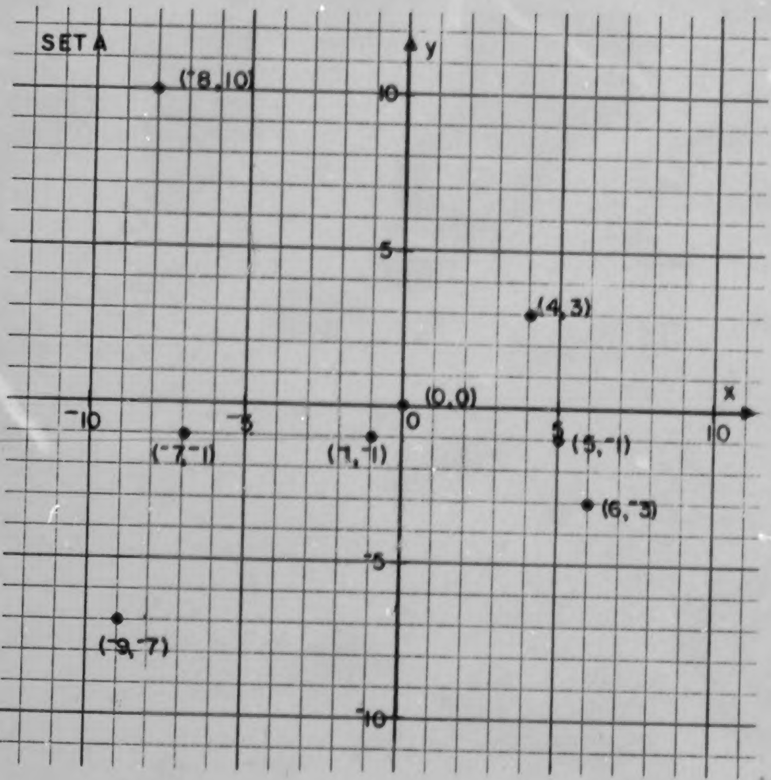


Answers to Exercises 1-4b

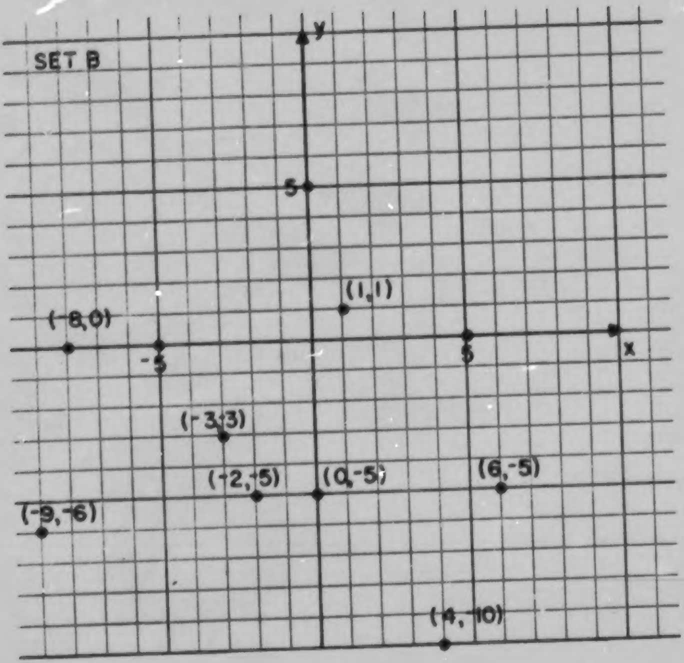
1.



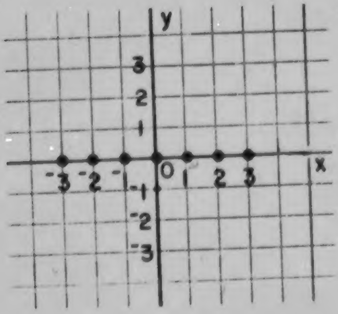
2.



2. con't



3. (a)

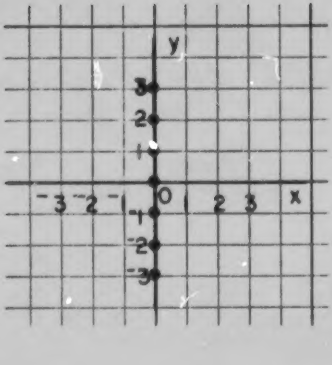


(b) yes

(c) y = 0

(d) no

4. (a)



(b) yes

(c) x = 0

(d) no

Answers to Class Exercises 1-4

1. (a) I

(b) IV

(c) II

(d) III

(e) I

(f) IV

(g) III

2. (a) I

(b) III

(c) II

(d) IV

3. (a) On the Y-axis, but not at the origin.

(b) On the X-axis, but not at the origin.

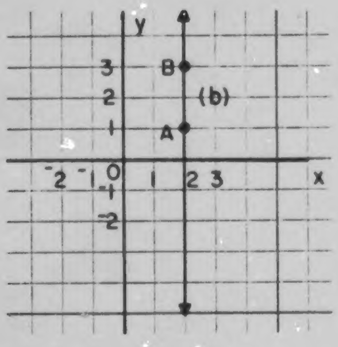
(c) At the origin

4. The quadrants are defined by the intersection of half-planes which do not contain the axes. Hence, their intersection does not contain the axes.

[pages 26-29]

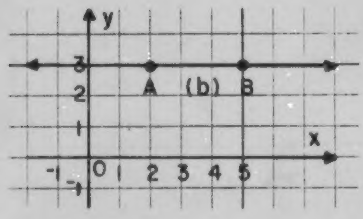
Answers to Exercises 1-4c

1.



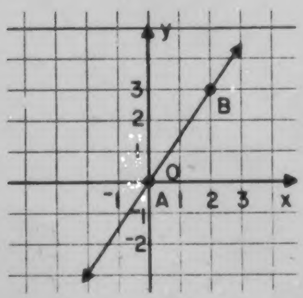
(c) the Y-axis

2.



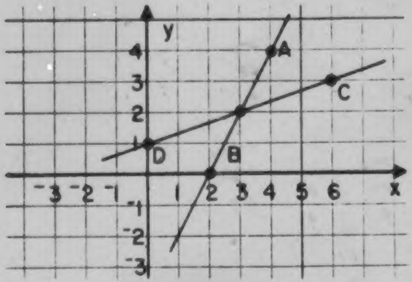
(c) the X-axis

3.



(c) neither axis.
AB is an oblique line.

4.



(e) {(3,2)}

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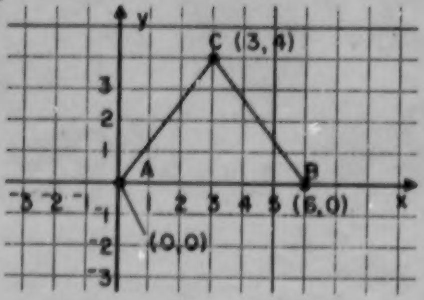
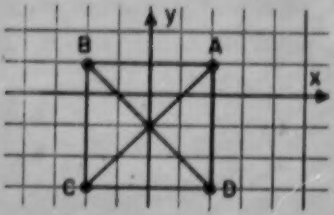
5.



(c) isosceles

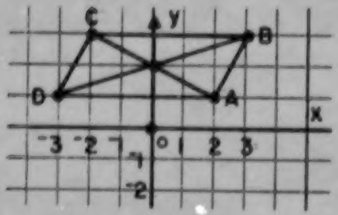
6.



(c) yes

(e) (0, $^-1$)

7.



(c) parallelogram

(e) (0, 2)

1-5. Graphs

In sketching the graphs of conditions (expressed by equations), the students are carrying the association of geometric ideas (points and lines) and arithmetic ideas (numbers, number pairs, and equations) one step further. The use of a coordinate system in the plane provides for an association of a line (a set of points) with an equation and an equation with a set of points. In this section, only the simplest cases of equations are considered. Until students have learned more about multiplication and subtraction of rational numbers (both positive and negative) we are limited to conditions expressed by equations of the types $x = a$, $y = a$, $y = x + a$ and $x = y + a$, where a is a rational number. Even in this small beginning it is not too early for students to think about the two-way association of equation to line and line to equation. The teacher may wish to ask students to sketch graphs of equations like $y = x + 2$ and $y = x + {}^-3$ at this time.

In this section we have chosen to talk about the graphs of a set of points described by a condition, such as $x = a$. Since properties of equations will be considered in Chapter 2, it has seemed better to talk about conditions on the coordinates of points, such as $y = x$, in an introductory treatment.

In addition to graphing equations of lines, pupils should be able to graph some inequalities. The graphs of inequalities in this section are all half-planes. The teacher may wish to review ideas associated with half-planes in the seventh grade course. It should present no special difficulty to pupils to recognize that all points above the line of the equation $y = 3$, have coordinates which satisfy the inequality $y > 3$.

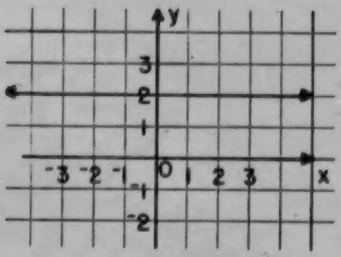
Answers to Class Exercises 1-5

1. $P(3,5)$, $M({}^-1,4)$, $T(1,2)$, $N(4,6)$, $A({}^-4,0)$, $B(0,6)$, $F({}^-6,6)$.
2. $H({}^-4,{}^-2)$ does satisfy the condition $y > x$.
   C and R also "belong" to the condition $y > x$.
   There is no difference between $y > x$ and $x < y$.
3. The points whose coordinates satisfy the condition $y > x$ all lie in the half-plane above the line whose condition is $y = x$.
4. The points whose coordinates satisfy $y < x$ lie in the half-plane below the line whose condition is $y = x$.

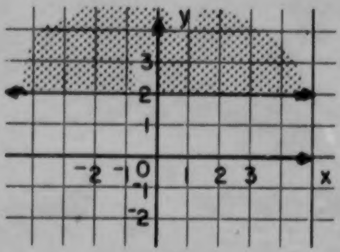
5. These points are in the half-plane below the line whose condition is $y = x$.
The points whose coordinates satisfy $x = y$ lie on the line that determines the half-planes and this line is not in either half-plane. (Refer to the definition of a half-plane given in Volume I.)

Answers to Exercises 1-5

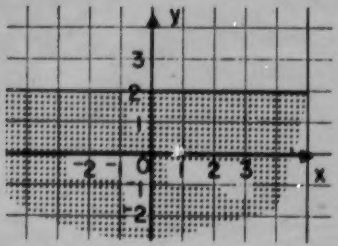
1. (a) $y = 2$



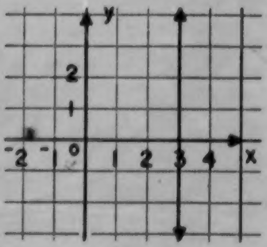
(b) $y > 2$



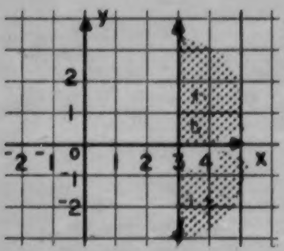
(c) $y < 2$



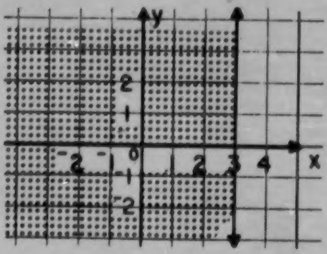
(d) $x = 3$



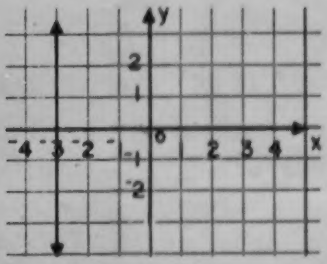
(e) $x > 3$



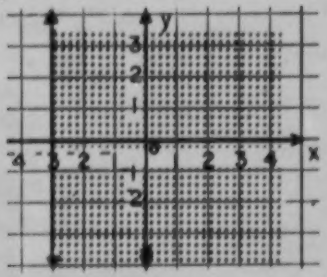
(f) $x < 3$



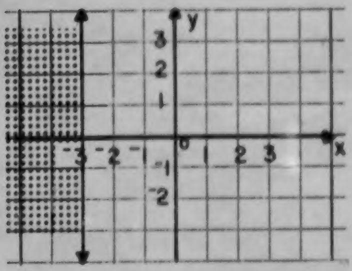
(g) $x = {}^{-}3$



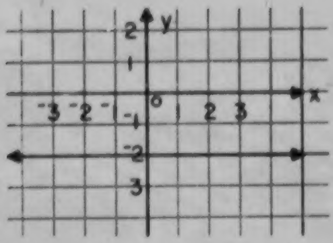
(h) $x > {}^{-}3$



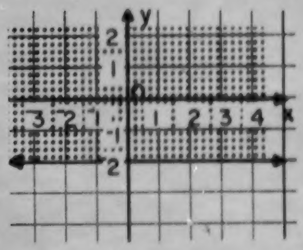
(i) $x < {}^{-}3$



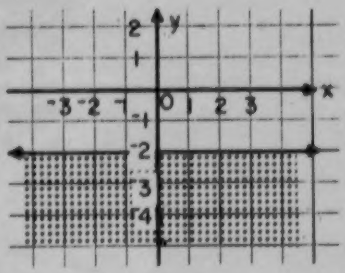
(j) $y = {}^{-}2$



(k) $y > {}^{-}2$



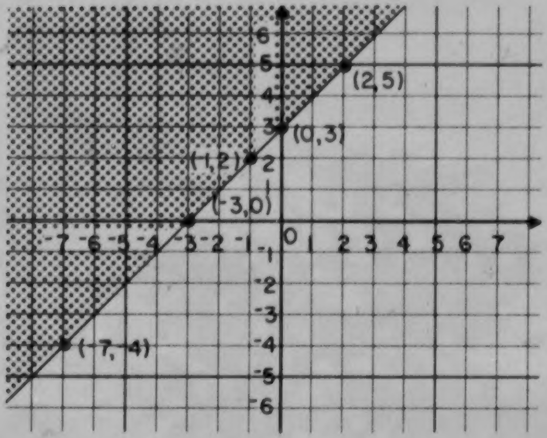
(l) $y < {}^{-}2$



(m) the graph of (a) is on the line two units above and parallel to the X-axis.
the graph of (b) is on the half-plane above the graph of $y = 2$.
the graph of (c) is on the half-plane below the graph of $y = 2$.

(n) the graph of (d) is on the line three units to the right of the Y-axis.
the graph of (e) is on the half-plane to the right of the graph of $x = 3$.
the graph of (f) is on the half-plane to the left of the graph of $x = 3$.

2. (a,b,c)



1-6. Multiplication of Rational Numbers

In this chapter the treatment of all topics is intended to be quite informal. Ideas are related to earlier study by the student so that conclusions may appear to the student to be reasonable ones. There is no proof of the conclusions. In multiplication, students are first asked to recall properties of a set of multiples of a whole number or of a multiplication table. Here are several examples: $5 \cdot 0 = 0$; $5 \cdot 1 = 5$; $5 \cdot 2 = 10$; $5 \cdot 3 = 15$; $5 \cdot 4 = 20$; etc. Each product is 5 less than the one which follows it. Hence, it is reasonable to conclude that if the number by which 5 is multiplied decreases by 1, the product decreases by 5. If we use integers to the left of 0, in order, the products can be expected to be $^{-}5$, $^{-}10$, $^{-}15$, etc. This leads us to the conclusion that $5 \cdot {}^{-}1 = {}^{-}5$, $5 \cdot {}^{-}2 = {}^{-}10$, $5 \cdot {}^{-}3 = {}^{-}15$, etc.

It is suggested that the completion of the multiplication table in Section 1-6 be done in class. The development is begun in the text by completing the "3 column" and the "3 row". Further development should follow with the "2 column" bottom to top, then the "2 row" right to left. The "1 column" and the "1 row" should then be developed. At this time it should be pointed out that the positive numbered columns read from the bottom to the top are all decreasing, but that the negative numbered columns are increasing in the same direction. With this information the students should inductively decide what should be used to complete the four cells at the upper left. Teachers may wish to extend the table in the negative direction and also to discuss the table with respect to some of the ideas in Volume I, Chapter 12.

This approach enables us to avoid the difficulties involved with negative multipliers, whether or not the number line is used. A second approach to multiplication uses the commutative, associative and distributive properties. Also there is no objection to the use of a number line approach (indeed, this appeared in Section 1-3 in the exercises) either as a supplementary method or as the principal method.

Teachers may also find it helpful to give other interpretations, such as, gains and losses in games, and running a film backward.

Full advantage should be taken of the opportunity to provide motivation for multiplication in pointing out that our ability to plot graphs of equations will be seriously limited unless we can find products of which at least one factor is a negative number.

The informal treatment cannot be expected to provide for immediate recall of all the facts and rules of operations with the rational numbers. The basic understandings and the reasonableness of the rules should be accepted. The treatment given here is an introduction to a topic which will be useful in Chapter 2 and other parts of the course, and particularly in later work in mathematics. Pupils should be allowed sufficient time to do the work. Perhaps a few students will not be able to give products automatically after studying this chapter.

There may be more routine "drill" exercises than the teacher will wish to assign. Some of these, particularly the last two, Problems 16 and 17, may be used for class discussion.

Answers to Exercises 1-6

1. The products increase in the 1-row, 2-row, and 3-row as we move to the right.
2. The products decrease, as we move down, in the ¯2 column and the ¯1 column.
3. ¯28, ¯21, ¯14, ¯7, 0, 7, 14, 21, 28, 35, 42
4. 20, 16, 12, 8, 4, 0, ¯4, ¯8, ¯12, ¯16, ¯20
5.

	¯3	¯2	¯1	0	1	2	
¯5	15	10	5	0	¯5	¯10	
¯4	12	8	4	0	¯4	¯8	
¯3	9	6	3	0	¯3	¯6	
¯2	6	4	2	0	¯2	¯4	
¯1	3	2	1	0	¯1	¯2	
0	0	0	0	0	0	0	
1	¯3	¯2	¯1	0	1	2	

6. (a) (¯2)(1) = ¯2, (1)(¯2) = ¯2, therefore (¯2)(1) = (1)(¯2)
   (b) (¯3)(0) = 0, (0)(¯3) = 0, therefore (¯3)(0) = (0)(¯3)
   (c) (¯2)(¯3)= 6, (¯3)(¯2) = 6, therefore (¯2)(¯3) =(¯3)(¯2)

(d) $(^{-}1)(^{-}3) = 3$, $(^{-}3)(^{-}1) = 3$, therefore
$(^{-}1)(^{-}3) = (^{-}3)(^{-}1)$

7. $(^{-}2)[(^{-}1)(5)] = (^{-}2)(^{-}5) = 10$, $[(^{-}2)(^{-}1)](5) = (2)(5) = 10$,
therefore $(^{-}2)[(^{-}1)(5)] = [(^{-}2)(^{-}1)](5)$

8. (a) $(^{-}4)(3) + (^{-}4)(8) = (^{-}12) + (^{-}32) = {}^{-}44$
$^{-}4(3 + 8) = {}^{-}4(11) = {}^{-}44$ therefore
$(^{-}4)(3) + (^{-}4)(8) = {}^{-}4(3 + 8)$

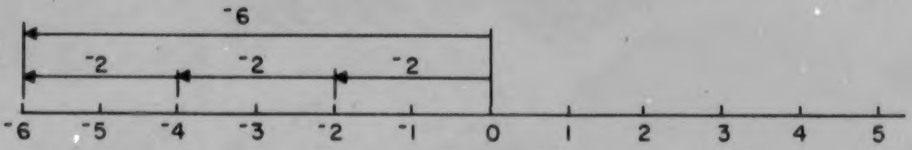
(b) $(^{-}2)(^{-}3) + (^{-}2)(6) = 6 + (^{-}12) = {}^{-}6$
$^{-}2(^{-}3 + 6) = (^{-}2)(3) = {}^{-}6$ therefore
$^{-}2(^{-}3 + 6) = (^{-}2)(^{-}3) + (^{-}2)(6)$

(c) $(^{-}10)(^{-}8) + (^{-}10)(^{-}1) = 80 + 10 = 90$
$^{-}10[(^{-}8) + (^{-}1)] = {}^{-}10(^{-}9) = 90$ therefore
$^{-}10[(^{-}8) + (^{-}1)] = (^{-}10)(^{-}8) + (^{-}10)(^{-}1)$

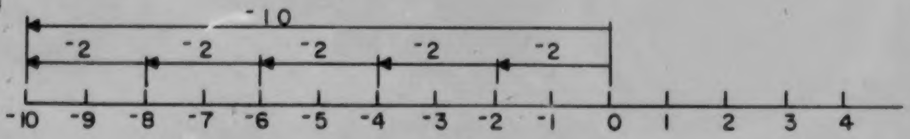
9. (a) 0
(b) $^{-}8$
(c) $^{-}20$
(d) $^{-}24$
(e) $^{-}34$
(f) $^{-}245$
(g) 54
(h) 600
(i) 903
(j) 0.84
(k) $^{-}60$
(l) 576
(m) $^{-}66$
(n) $^{-}(\frac{160}{3})$
(o) $^{-}16$

10. (a) $^{-}4$
(b) $^{-}5$
(c) $^{-}11$
(d) $^{-}8$
(e) $^{-}77$

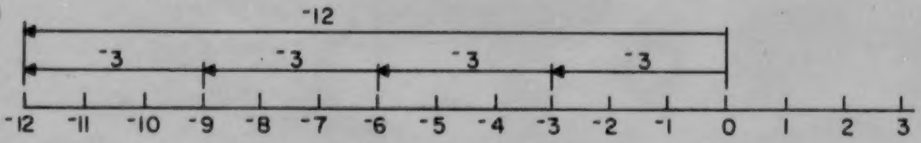
11. (a)


$^-6$
$^-2$
$^-2$
$^-2$
$^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5

(b)


$^-10$
$^-2$
$^-2$
$^-2$
$^-2$
$^-2$
$^-10$ $^-9$ $^-8$ $^-7$ $^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4

(c)


$^-12$
$^-3$
$^-3$
$^-3$
$^-3$
$^-12$ $^-11$ $^-10$ $^-9$ $^-8$ $^-7$ $^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3

12. Since $(^-4)(3) = (3)(^-4)$, we find the product of (3) and $(^-4)$ on the number line. After we find this product on the number line we claim it equals $(^-4)(3)$ because of the commutative property of multiplication.

13. (a) 39 - yard line

(b) $45 + [(3)(^-2)]$ or $45 + (^-6)$

14. $^-9$

15. (a) $^-12$

(b) $^-15$

(c) $^-5$

(d) $^-10$

(e) 4

(f) 2

16. (a) $^-2$

(b) $^-3$

(c) $^-10$

(d) $^-4$

(e) 4

(f) $^-10$

(g) $^-1$

(h) 0

(i) $^-1$

(j) $^-6$

(k) $^-9$

(l) $^-6$

(m) 9

(n) $^-2$

(o) 10

(p) 2

17. (a) 60
(b) 12
(c) 15
(d) 42
(e) 23
(f) 300
(g) 40
(h) $^{-}40$
(i) $^{-}40$
(j) 42
(k) 60
(l) 110
(m) $^{-}192$
(n) 192
(o) 135
(p) $^{-}75$
(q) 0
(r) $^{-}16$
(s) 1000
(t) $^{-}60$
(u) 60
(v) 60
(w) $^{-}6$
(x) 16
(y) $^{-}27$
(z) 8

1-7. Division of Rational Numbers

In an informal introduction to negative numbers it seems appropriate to think of division only in terms of the inverse operation, multiplication. Similarly subtraction, by the additive method is the development chosen for this chapter. In the work of the remainder of this course the student will rarely, if ever, find it necessary to divide or subtract, using negative numbers. After studying this section and the next, the pupil should feel that he knows how to find the answers to such questions as, "What is the quotient of $^{-}5$ and 2?" and "What is the number x for which $8 - (^{-}7) = x$?" even though his response may be by no means automatic.

The teacher should make full use of the opportunity to relate multiplication and division and to review some of the basic ideas about rational numbers developed in Chapter 6 of Volume I. It is recommended that emphasis be placed on interpreting the question, "What is $^{-}6 \div {}^{-}2$?" as another way of asking, "By what number should $^{-}2$ be multiplied to obtain $^{-}6$?" Frequent early reference might well be made to what is known from arithmetic. We know $15 \div 3 = 5$ because $5 \cdot 3 = 15$.

It is also recommended that the equation be used extensively in this development. There are two points to be emphasized.

Find n if $^{-}5n = {}^{-}35$.

a. Since we know $^{-}5 \cdot 7 = {}^{-}35$, $n = 7$ in

$$^{-}5n = {}^{-}35.$$

b. If

$$^{-}5n = {}^{-}35$$

$$n = \frac{^{-}35}{^{-}5}$$ by our extended definition of rational numbers to include negative numbers.

$$\frac{^{-}35}{^{-}5} = 7, \text{ since } {}^{-}5 \cdot 7 = {}^{-}35.$$

In Exercises 1-7, Problems 8 through 15 provide for a development of a generalization about conditions under which the quotient is a negative or a positive number (or under which a fraction with negative numerator or denominator is a numeral for a negative number). It may be desirable to use these exercises for class discussion.

The teacher should notice that we do not talk about the "sign of a number" throughout the chapter. "Negative eleven" is a number, just as 193 is a number. Negative three-fourths, $^{-}(\frac{3}{4})$, is a number. As a number it does not have a numerator and a denominator. We may talk about the numerator and denominator of a fractional numeral for this negative number, such as $\frac{(^{-}3)}{4}$, or $\frac{3}{(^{-}4)}$.

In the last paragraph of this section there is one example involving the reciprocal of a negative number. This example is used as further illustrative material. We are interested often is the number, a, by which another number, b, must be multiplied to obtain the product, 1. The numbers a and b may be called multiplicative inverses.

Answers to Exercises 1-7

1. (a) $^{-}28$
   (b) $^{+}12$
   (c) $^{-}12$
   (d) $^{-}72$
   (e) 72
   (f) 735
   (g) $^{-}3$
   (h) 4
   (i) $\frac{2}{5}$

2. (a) $^{-}4$
   (b) $^{-}4$
   (c) $^{-}6$
   (d) $^{-}24$
   (e) $^{-}8$
   (f) $^{-}21$
   (g) $^{-}(\frac{3}{4})$
   (h) $^{-}10$
   (i) $^{-}(\frac{12}{25})$

3.

x	$\frac{1}{2}x$
4	2
2	1
0	0
$^{-}2$	$^{-}1$
$^{-}4$	$^{-}2$
$^{-}5$	$^{-}2\frac{1}{2}$
$^{-}6$	$^{-}3$

4.

x	$^{-}4x$
2	$^{-}8$
$\frac{3}{2}$	$^{-}6$
1	$^{-}4$
0	0
$^{-}(\frac{1}{2})$	2
$^{-}(\frac{3}{2})$	6
$\frac{3}{4}$	$^{-}3$

5. (a) $r = 5\frac{2}{3}$
(b) $r = {}^{-}2$
(c) $r = {}^{-}3$
(d) $r = 7$
(e) $r = \frac{3}{16}$
(f) $r = {}^{-}(\frac{1}{2})$
(g) $r = {}^{-}1\frac{1}{7}$
(h) $r = 1$
(i) $r = {}^{-}(\frac{1}{12})$

6. $P' = [\frac{1}{6},\ \frac{2}{3},\ 1,\ \frac{6}{5},\ {}^{-}1,\ {}^{-}(\frac{2}{3}),\ {}^{-}(\frac{3}{7})]$

7. (a) 2
(b) $^{-}5$
(c) $^{-}5$
(d) 5
(e) $^{-}5$
(f) 5
(g) 5
(h) 3
(i) $^{-}36$
(j) 21
(k) $^{-}22$
(l) 13
(m) $^{-}4$
(n) $^{-}25$
(o) 0
(p) 13
(q) $^{-}12$
(r) $^{-}6$

8. (a) $n = \frac{({}^{-}2)}{3} = {}^{-}(\frac{2}{3})$
(b) $n = \frac{2}{(-3)} = {}^{-}(\frac{2}{3})$

9. $\frac{({}^{-}2)}{3} = \frac{2}{(-3)}$

10. (a) $n = \frac{-6}{7} = {}^{-}(\frac{6}{7})$
(b) $n = \frac{6}{-7} = {}^{-}(\frac{6}{7})$

11. $({}^{-}6)n = 7, \quad 6n = {}^{-}7$

12. (a) $n = (\frac{92}{25}) = 3\frac{17}{25}$
(b) $n = \frac{92}{25} = 3\frac{17}{25}$
(c) $n = \frac{1}{3}$
(d) $n = \frac{1}{3}$

13. $\frac{^{-}92}{^{-}25}$

14. $25n = 92$ $\quad$ $^{-}25n = {}^{-}92$

15. (a) $bx = a$
    (b) either both positive or both negative
    (c) negative, positive

1-8. Subtraction of Rational Numbers

Some pupils will have first studied subtraction by the additive method but these pupils are almost certainly in the minority. Nevertheless all pupils should be familiar, from arithmetic, with the relationship involved and its importance. By the additive method in order to subtract 8 from 17 we think of what number added to 8 is 17. If the teacher uses this approach in this chapter, some review, using numbers from arithmetic, will be desirable.

The term additive inverse is first introduced in 1-3 and is used again in Problems 6 and 7. The teacher may choose to use additive inverses more extensively in class discussion. The use of this idea is often helpful to students as a tool in obtaining correct answers. It is essential in the chapter that the student make such generalizations as "to subtract $^{-}12$ is the same as adding 12."

Answers to Exercises 1-8

1. (a) 3
   (b) 0
   (c) $^{-}7$
   (d) 5
   (e) 10
   (f) 8
   (g) $^{-}45$
   (h) $^{-}67$
   (i) $^{-}28$

2. $\frac{11}{4}$, $\frac{11}{4} - {}^-(\frac{3}{4}) = \frac{7}{2}$, $\frac{11}{4} - \frac{7}{2} = {}^-(\frac{3}{4})$

3. (a) $x = {}^-3$

(b) $x = 11$

(c) $x = {}^-11$

(d) $x = 15$

(e) ${}^-(\frac{13}{6})$

(f) ${}^-(\frac{3}{2})$

(g) 5

(h) ${}^-(\frac{49}{12})$

4. (a) ${}^-5$

(b) 3

(c) ${}^-6$

(d) 6

(e) 3

(f) 3

(g) $\frac{1}{4}$

(h) ${}^-(\frac{1}{2})$

5. (a) ${}^-6$

(b) ${}^-5$

(c) 11

(d) 2

(e) ${}^-(\frac{6}{4}) = {}^-(\frac{3}{2})$

(f) 2

(g) $\frac{41}{3}$

(h) $\frac{117}{7}$

6. (a) ${}^-10$

(b) 100

(c) ${}^-(\frac{1}{2})$

(d) ${}^-(\frac{7}{9})$

(e) $\frac{8}{5}$

(f) $\frac{49}{51}$

7. (a) $({}^-4) + ({}^-2)$

(b) $({}^-6) + (1)$

(c) $8 + (3)$

(d) $({}^-11) + (13)$

(e) ${}^-(\frac{5}{4}) + {}^-(\frac{1}{4})$

(f) ${}^-(\frac{3}{2}) + (\frac{7}{2})$

(g) $\frac{35}{3} + (2)$

(h) $\frac{75}{7} + (6)$

[pages 48-49]

8. (a) $^-7$
(b) $^-2$
(c) 4
(d) 10
(e) $^-10$
(f) $^-6$
(g) $^-11$
(h) 12
(i) 2
(j) 12
(k) $^-7$
(l) 11
(m) $^-7$
(n) 13

9. (a)

x	$2x$	$2x - 3$
$^-1$	$^-2$	$^-5$
2	4	1
$^-4$	$^-8$	$^-11$
0	0	$^-3$
$^-7$	$^-14$	$^-17$
$^-9$	$^-18$	$^-21$

(b)

x	^-2x	$^-2x - {}^-1$
$^-1$	2	3
0	0	1
3	$^-6$	$^-5$
4	$^-8$	$^-7$
5	$^-10$	$^-9$
$^-2$	4	5

Sample Questions

True-False

Read each of the following statements. Decide whether each statement is True or False. If the statement is true, write "True" in the space provided. If a statement is false, write "False" in the space provided.

T 1. We can associate rational numbers with points on a line.

F 2. The point named by the ordered pair (3,2) is the same as the point named by (2,3).

F 3. The point ($^-4$, $^-1$) is located in Quadrant II.

T 4. The sum of positive two and negative two is zero.

T 5. There is no greatest number on the number line.

F 6. All negative rational numbers are associated with points on the number line to the right of zero.

T 7. The point on the number line associated with $^{+}2$ and the point on the number line associated with $^{-}2$ are equidistant from the point on the number line associated with 0.

F 8. The rational number $^{-}3$ is greater than the rational number $^{-}2$.

T 9. The graph of a condition is the set of points in the coordinate plane described by that condition.

F 10. The product and quotient of two negative numbers is a negative number.

Completion

Complete each of the following statements by supplying the word or words which makes the statement a true statement. Place your answers in the spaces provided at the left of each statement.

$^{-}7$ 1. The y-coordinate in the ordered pair $(3, {}^{-}7)$ is ________.

third 2. The ordered pair $({}^{-}4, {}^{-}2)$ names a point in the coordinate plane located in the ________ quadrant.

$^{-}2$ 3. The sum of $^{+}4$ and $^{-}6$ is ________.

$^{+}10$ 4. The difference $({}^{+}4) - ({}^{-}6)$ is ________.

$^{-}24$ 5. The product of $^{+}4$ and $^{-}6$ is ________.

$^{-}(\frac{2}{3})$ 6. The quotient of $({}^{+}4) \div ({}^{-}6)$ is ________.

graph 7. The set of points in the coordinate plane described by a given condition is called the ________ of that condition.

10 8. If $^{-}5$ is subtracted from $^{+}5$, the difference is ________.

opposites or additive inverses 9. On a number line, the rational numbers assigned to two points equally distant from the origin are called ____________ of each other.

right 10. On a number line, $^{-}6$ is greater than $^{-}7$ because it is assigned to a point which is to the ________ of the point to which $^{-}7$ is assigned.

Multiple Choice

Read each statement carefully. Select the letter of the answer you think is correct. Place the letter in the answer space provided at the left of each statement.

A 1. Numbers which we associate with points on the number line to the left of 0 are called:
(A) Negative numbers (B) Positive numbers

B 2. Each point in the coordinate plane has how many numbers associated with it?
(A) one (B) two

B 3. The sum of $^{-}3$ and $^{-}6$ is:
(A) $^{+}9$ (B) $^{-}9$

A 4. The difference $(^{-}3) - (^{-}6)$ equals:
(A) $^{+}3$ (B) $^{-}3$

B 5. The product of $^{-}3$ and $^{-}6$ is:
(A) $^{-}18$ (B) $^{+}18$

A 6. The quotient of $(^{-}3) \div (^{-}6)$ equals:
(A) $^{+}(\frac{1}{2})$ (B) $^{-}(\frac{1}{2})$

A 7. $4 \cdot (1 - 2)$ equals:
(A) $^{-}4$ (B) $^{+}4$

B 8. The difference $(^{+}3) - (^{+}2)$ is the same as:
(A) $^{+}3 - (^{-}2)$ (B) $^{+}3 + (^{-}2)$

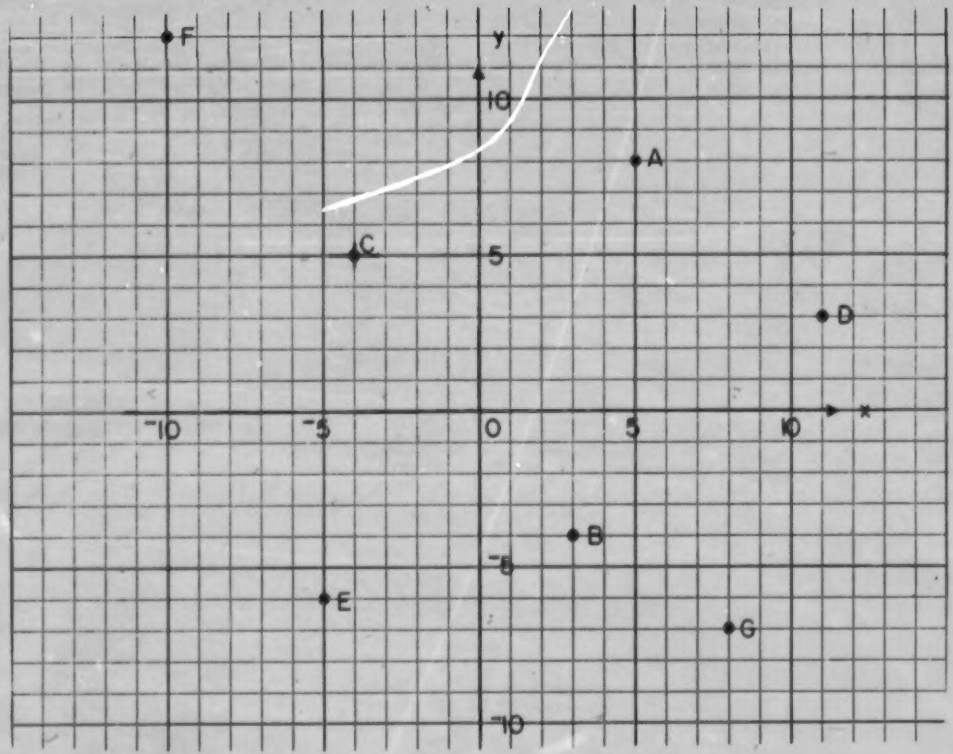
B 9. The point in the coordinate plane named by the ordered pair $(3, ^{-}3)$ is located in
(A) Quadrant III (B) Quadrant IV

A 10. The graph of the condition $y > x$ lies:
(A) in a half-plane (B) on a line

Matching

Study the items in Column A. Select the items from Column B which are best matched with the items listed in Column A. Place the letter of your choice from Column B in the answer spaces next to the items in Column A.

		Column A		Column B
h	1.	The sum of $^{-}6$ and $^{+}3$	(a)	points on a half-plane
i	2.	The difference: $(^{-}6) - (^{+}3)$	(b)	1
k	3.	The product of $(^{-}6)$ and $(^{+}3)$	(c)	$^{+}3$
i	4.	The quotient $(^{-}6) \div (^{+}3)$	(d)	0
g	5.	A point located in Quadrant II	(e)	points on a line
f	6.	An ordered pair described by the condition $x = 2$	(f)	(2, 4)
a	7.	The graph of $y > x$	(g)	$(^{-}3, ^{+}4)$
c	8.	The opposite of $^{-}3$	(h)	$^{-}3$
d	9.	The sum of a number and its opposite	(i)	$^{-}2$
b	10.	The product of 3 and $\frac{1}{3}$	(j)	$^{-}9$
			(k)	$^{-}18$
			(ℓ)	18
			(m)	(4, 2)
			(n)	$^{-}1$



In the diagram above points A through G have been plotted. Select the ordered pairs from Column b which are correctly associated with the plotted points.

	Column a		Column b
4	A(,)	1.	(¯5, ¯6)
7	B(,)	2.	(¯4, 5)
2	C(,)	3.	(¯3, 4)
8	D(,)	4.	(5, 8)
1	E(,)	5.	(4, ¯5)
12	F(,)	6.	(¯11, 3)
10	G(,)	7.	(3, ¯4)
		8.	(11, 3)
		9.	(10, ¯12)
		10.	(8, ¯7)
		11.	(5, 6)
		12.	(¯10, 12)

Other Sample Questions

1. k, ℓ, m, and n are rational numbers.

   (a) $k + \ell = m$. To what number is $m - \ell$ equal?

   (b) $k \cdot \ell = m$. Express k in terms of ℓ and m.

2. Supply the missing number to make a true statement.

   (a) $4 + (^-5) + (\quad) = 0$

   (b) $^-13 + (\quad) + 17 = 0$

   (c) $(\quad) + (^-13) + (^-4) = 0$

   (d) $^-8 \cdot (\quad) = {}^-16$

   (e) $^-15 - (\quad) = 8$

3. Sketch a number line and mark the points associated with the numbers

   $^-(\frac{1}{2})$, 6, $^-3$, $\frac{9}{2}$, and $^-(\frac{4}{3})$

4. The average of five numbers is obtained by adding the numbers and dividing the sum by 5. Find the average of the numbers: $^-8$, $^-3$, $^-1$, 0, 10.

5. Complete the table for $y = {}^-3x$

x	$^-2$	$^-1$	0	1	
^-3x					$^-12$

6. What condition is satisfied by the ordered pairs $(3, {}^-3)$, $(2, {}^-2)$, $(^-1, 1)$, $(0, 0)$, $(^-28, 28)$?

7. Graph the inequality $x > {}^-2$.

8. At the end of a game consisting of 5 rounds of play, Bill's score is 23 and Jack's score is $^-15$. What is the difference in their scores?

9. The vertices of a trapezoid are at the points $(^-4, 3)$, $(0, 0)$, $(3, 0)$, and $(7, 3)$. Plot these points. What are the lengths of the parallel sides of the trapezoid?

10. Draw a sketch to show the addition on a number line of $^-3$ and $^-5$.

11. What is the coordinate of the point on the number line midway between the points given?

(a) (4) and $(^-6)$ (b) $(^-10)$ and $(^-1)$

Answers to Other Sample Questions

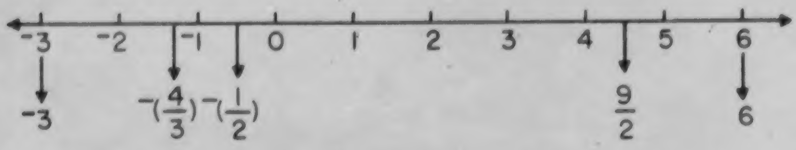
1. (a) $m - \ell = k$ (b) $k = \frac{m}{\ell}$

2. (a) 1 (d) 2

(b) $^-4$ (e) $^-23$

(c) 17

3.



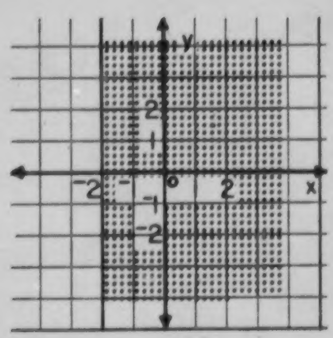
4. $$\frac{(^-8) + (^-3) + (^-1) + 0 + 10}{5} = ^-\left(\frac{2}{5}\right)$$

5.

x	$^-2$	$^-1$	0	1	4
^-3x	6	3	0	$^-3$	$^-12$

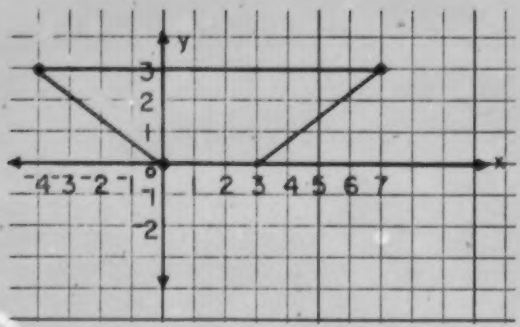
6. $y = {}^-x$ or $x = {}^-y$

7. Graph of $x > {}^-2$



8. 38

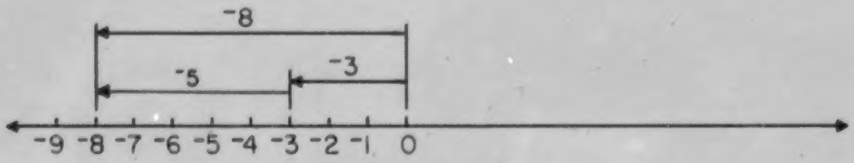
9.


y
x
3
2
1
0
⁻1
⁻2
⁻4 ⁻3 ⁻2 ⁻1
1 2 3 4 5 6 7

(a) longest side is 11

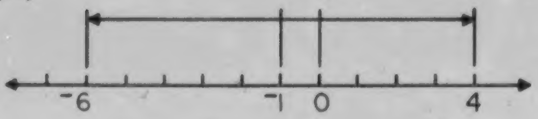
(b) shortest side is 3

10.


⁻8
⁻5
⁻3
⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0

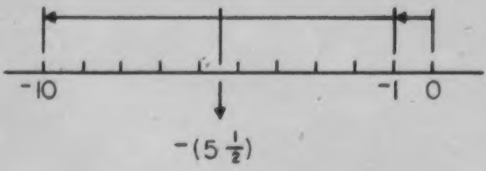
11. (a)

$$\frac{4 + (^-6)}{2} = \frac{^-2}{2} = {}^-1$$



(b)

$$\frac{(-10) + (^-1)}{2} = \left(\frac{^-11}{2}\right) = {}^-(5\tfrac{1}{2})$$



Chapter 2

EQUATIONS

Introduction

It is expected that 15 days will be required for this chapter. This should allow sufficient time for covering the material included in this chapter, a day for testing, and a day for discussion and review following the test.

Tell the pupils that they cannot read mathematics like a novel. They should have pencil and paper handy while reading the text, and they should work out the answers to any questions in a paragraph before going on to the next. They cannot absorb ideas like sponges. They must take an active part in the learning process. Although the authors of this chapter have tried to keep the language at the junior high school level, this is not always possible due to the material covered. Careful reading on the part of the pupil, coupled with participation in carefully prepared class discussions should lead the pupil to good understanding of the ideas covered in this chapter.

At this time, we are trying to convey the meanings of concepts related to solving equations and to give the pupils some of the technical vocabulary he will need later. We are not trying, in this chapter, to develop great skill in solving equations, nor do we expect complete mastery of the subject. These goals are left to the formal, systematic instruction in algebra in the ninth grade. Hence, we have avoided introducing such terms as "variable" and "constant." We aimed at the introduction of these ideas but felt it desirable to delay the introduction of the vocabulary until a more formal approach to equations is used.

It is advisable, therefore, to avoid dwelling at great length on this chapter. Pupils will have ample practice with equations later. Do not expect the pupils to achieve complete mastery of topics normally covered in the ninth grade.

2-1. Writing Number Phrases

The first part of this section is intended to interest pupils in the solution of equations. Terminology such as, "number phrase," "number sentence," etc., should not be overemphasized. For instance, there is no point in exercises in which the pupil is asked to distinguish between a number phrase and a number sentence. The only purpose in the terminology is to have a name for what we are writing about.

The chief purpose of this section is to give the pupil practice in expressing number phrases in mathematical terms--that is, in terms of letters and numbers. Also translation in the opposite direction is important.

Some teachers may also introduce a little practice in translating simple, real-life situations into the forms of equations and number phrases. Not much of this should be done at this time, however, since the phrases must be simple. It is important to write explicitly such statements as:

Let x = the number of years in Ann's age.

Stating exactly what the letter stands for is very important.

It may be necessary to review the customary way of writing the product of an unknown and a counting number as $3x$ instead of $3 \cdot x$ or $(3) \cdot (x)$. Practice on such things as $3x + 2x = 5x$ occurs later in this chapter.

The class should be given sufficient practice in translating number phrases given in words, into phrases in symbolic language, and in translating symbols into words. Great care must be used with the terms "difference" and "quotient." Since the commutative property does not hold for the operations of subtraction and division, the expression, "the difference of 7 and 8" may not have any meaning. Should this be interpreted as $8 - 7$ or $7 - 8$? Having been introduced to negative numbers in Chapter 1, the pupil should learn to be very careful about expressions which may not have been questioned previously but which now appear ambiguous.

The class exercises may be done partly as a cooperative effort in class and partly as supervised study so that the students may become fairly confident before doing problems entirely on their own.

Answers to Class Exercises 2-1

1. (a) $x + 5$
   (b) $x - 3$
   (c) $8x$
   (d) $\frac{1}{4}x$ or $\frac{x}{4}$
   (e) $x + 10$
   (f) $7x$
   (g) $x - 11$
   (h) $\frac{x}{2}$
   (i) $x - 6$
   (j) $x - 9$

2. (a) 17
   (b) 9
   (c) 96
   (d) 3
   (e) 22
   (f) 84
   (g) 1
   (h) 6
   (i) 6
   (j) 3

3. Pupils will undoubtedly write different translations for each of these phrases. Teachers should be primarily concerned with correct order of numbers when different versions are presented by pupils.

   (a) The number x increased by 1
   (b) The number x decreased by 3
   (c) The number x multiplied by 2
   (d) The number 18 divided by x
   (e) The number x multiplied by 4
   (f) The number x added to $^{-}6$

4. (a) 7
(b) 3
(c) 12
(d) 3
(e) 24
(f) 0

5. (a) $^{-}1$
(b) $^{-}5$
(c) $^{-}4$
(d) $^{-}9$
(e) $^{-}8$
(f) $^{-}8$

6. (a) $d + 5$
(b) $d + (d + 5) = 2d + 5$

7. (a) $10d$
(b) $\frac{1}{4}q$ or $\frac{q}{4}$
(c) $3y$
(d) $5(n + 1)$
(e) $12(f - 3)$

8. (Here various letters can be used. We use the letter n.)
(a) $n + 4$
(b) $n + 2n = 3n$
(c) $n + 7$
(d) $n - 5$
(e) $5 - n$
(f) $9n$
(g) $\frac{n}{10}$
(h) $\frac{10}{n}$
(i) $2n - n = n$
(j) $\frac{3n}{2n} = \frac{3}{2}$

It is not necessary to simplify the results but some students may do it.

Answers to Exercises 2-1

1. (a) $6 + a$
(b) $8b$
(c) $8c + 1$
(d) $8d - 3$
(e) $\frac{8e}{4}$
(f) $2f + 3$
(g) $5(g + 2)$
(h) $7h - 10$
(i) $\frac{12}{i + i}$
(j) $(j + 3)(j + 4)$

[pages 54-56]

2. (a) 3
(b) $^{-}24$
(c) $^{-}23$
(d) $^{-}27$
(e) $^{-}6$
(f) $^{-}3$
(g) $^{-}5$
(h) $^{-}31$
(i) $^{-}6$
(j) 0

3. (a) Five more than two times a certain number
(b) Three times a certain number and the result subtracted from six
(c) A number decreased by one and the result multiplied by 7
(d) Five decreased by a number and the quantity divided by two
(e) Fifteen increased by two times a number

4. (a) 15
(b) $^{-}5$
(c) 5
(d) 3

5. (a) 6
(b) 9
(c) 3
(d) $^{-}9$

6. (a) $x + 3$
(b) $10y$
(c) $25q$
(d) $y + 5$
(e) $z - 6$
(f) $g + 2g = 3g$
(g) $n - 6$
(h) $4k + (k + 20) + k$
(i) $5k + 10(k + 1)$

7. $n + (n + 1) = 2n + 1$

8. Let n be an odd number; then the sum of two consecutive odd numbers of which n is the first is: $n + (n + 2)$.

9. Let n be the first of three consecutive even numbers. Then the sum of the three consecutive even numbers will be

$$n + (n + 2) + (n + 4).$$

Some students may prefer to let 2n stand for the first of the even numbers.

10. If n is the first of the consecutive multiples of ten, the sum of the two will be: $n + (n + 10)$. Some students may prefer to let 10n be the first of the consecutive multiples of 10. In this case the sum of the two will be $10n + (10n + 10) = 20n + 10$. Another form of this result is: $10(2n + 1)$.

2-2. Writing Number Sentences

The objective of this section is to give the student instruction and practice in writing number sentences--that is, equations and inequalities. Again the terminology should not be overemphasized, but it is important that statements should be clean-cut so that the student may know exactly what is meant by a solution of an equation or inequality.

The connection between the ideas here and the use of formulas is important--at least for the formulas with which the student is already familiar.

It is not intended that, with the study of this chapter, the student will completely master the solution of equations. That is left to the ninth grade. For the most part, only linear equations are considered.

The comments made about class exercises for the previous section apply here as well.

The section on graphs of solution sets attempts to clarify the meaning of solution set. While we feel that such representation is helpful at various points where fundamental ideas are presented, the drawing of graphs on the number line should not become an end in itself. As soon as the ideas become clear the graphs may be discontinued.

[pages 57-60]

The portion of the section before the last exercise is intended as a kind of summary.

There are places in this section where solutions are called for. Here the pupils should find the solutions obvious. This should develop the idea of what is meant by a solution and should lead into methods, discussed in the next section, of arriving at a solution where inspection methods fail.

Answers to Class Exercises 2-2a

1. (a) $x = 2$
   (b) $y > 2$
   (c) $k = {}^{-}28$
   (d) $m = 6$
   (e) $s < 6$
   (f) $t \neq {}^{-}5$

2. (a) $x = 9$
   (b) $y > 9$
   (c) $n = 7$
   (d) $x = 9$
   (e) $p = 14$
   (f) $x < 14$

3. (a) $b = 3$
   (b) $a \neq 3$
   (c) $w = 7$
   (d) $m < 7$
   (e) $x = {}^{-}1$
   (f) $y = {}^{-}8$

4. (a) $n = 6$
   (b) $a < 6$
   (c) $k = {}^{-}16$
   (d) $d > 18$
   (e) $h = {}^{-}15$
   (f) $s = 21$

5. $p = 2 \cdot 7 + 2 \cdot 4 = 14 + 8 = 22$

6. $A = \frac{1}{2}(14)(7) = 49$

Answers to Exercises 2-2a

1. (a) $x + 5 = 13$
   (b) $x - 3 = 7$
   (c) $8x = 24$
   (d) $\frac{x}{4} = 9$
   (e) $x + 10 = 21$
   (f) $7x = {}^{-}35$
   (g) $x - 11 = {}^{-}5$
   (h) $x - 6 = 15$
   (i) $\frac{x}{2} = {}^{-}7$

2. (a) $\{8\}$
   (b) $\{10\}$
   (c) $\{3\}$
   (d) $\{36\}$
   (e) $\{11\}$
   (f) $\{{}^{-}5\}$
   (g) $\{6\}$
   (h) $\{21\}$
   (i) $\{{}^{-}14\}$

3. (a) $x + 2 > 4$
   (b) $5x < 10$
   (c) $\frac{x}{7} > 2$
   (d) $x - 3 > 6$
   (e) $x - 5 < 13$
   (f) $3x > {}^{-}9$

4. (a) The set of all numbers greater than 2
   (b) The set of all numbers less than 2
   (c) The set of all numbers greater than 14
   (d) The set of all numbers greater than 9
   (e) The set of all numbers less than 18
   (f) The set of all numbers greater than $^{-}3$

5. (a) Two more than a certain number is equal to five.
   (b) The sum of a certain number and negative three is seven.
   (c) A number multiplied by two is equal to negative ten.
   (d) A number added to the opposite of five is greater than nine.
   (e) The product of five and a number is less than fifteen.
   (f) The sum of seven and the opposite of a number is two.

(g) The sum of a number and negative three is less than four.

(h) A number divided by 3 is greater than nine.

(i) The sum of a number and the opposite of seven is negative two.

(j) A number divided by $^{-}30$ is equal to six.

6. (a) {3}

(b) {10}

(c) {$^{-}5$}

(d) The set of all numbers greater than 14

(e) The set of all numbers less than 3

(f) {5}

(g) The set of all numbers less than 7

(h) The set of all numbers greater than 27

(i) {5}

(j) {$^{-}180$}

7. $A = 225$ (square inches)

8. $i = 135$ (dollars)

9. $c = 62\frac{6}{7}$ (inches) or 62.8 (inches)

10. $d = 585$ (miles)

*11. $p = 142.50$ (dollars)

*12. $A = 531\frac{1}{7}$ (square feet) or 530.66 (square feet)

*13. (a) $v = 11$ (cubic feet)

(b) Capacity is $82\frac{1}{2}$ gallons

*14. (a) 32° F (c) 98.6° F

(b) 212° F

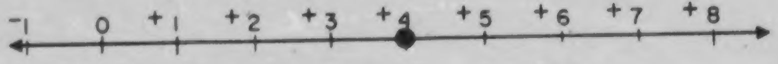
Answers to Class Exercises 2-2b

1. $\{2\}$. $(x = 2)$
2. $\{{}^-3\}$. $(x = {}^-3)$
3. All numbers greater than 4. $(x > 4)$
4. All numbers except zero. $(x \neq 0)$
5. $\{0, 1, 2, 3, 4, \ldots\}$. x is a non-negative integer.
6. $\{1, 2, 3, 4, \ldots\}$. x is a counting number or x is a positive integer.
7. All numbers less than negative one. $(x < {}^-1)$
8. All real numbers.
9. All numbers greater than negative two and less than 0. ${}^-2 < x < 0$ or, ${}^-2 < x$ and $x < 0$.
10. All integers.

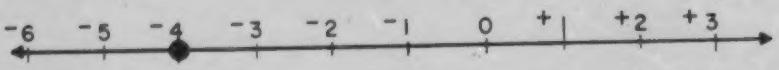
Answers to Exercises 2-2b

1. (a) $\{4\}$
   (b) $\{{}^-4\}$
   (c) $\{3\}$
   (d) {all numbers less than 1}
   (e) {all numbers greater than 5}
   (f) $\{{}^-5\}$
   (g) {all numbers less than 5}
   (h) {all numbers less than 3}

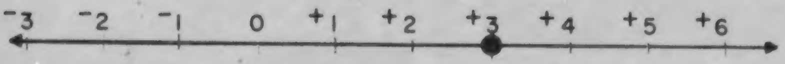
2. (a)



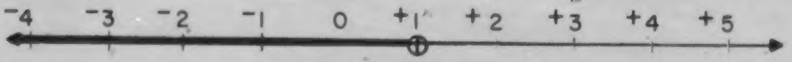
(b)



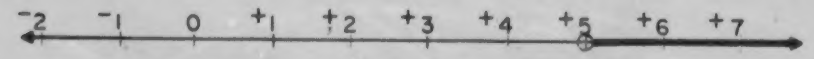
(c)


-3
-2
-1
0
+1
+2
+3
+4
+5
+6

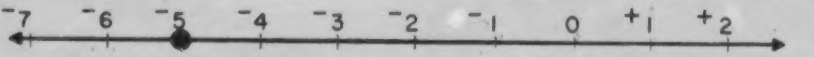
(d)


-4
-3
-2
-1
0
+1
+2
+3
+4
+5

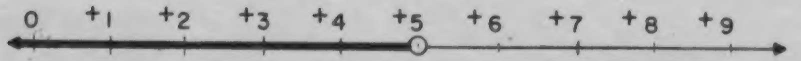
(e)


-2
-1
0
+1
+2
+3
+4
+5
+6
+7

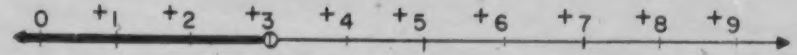
(f)


-7
-6
-5
-4
-3
-2
-1
0
+1
+2

(g)


0
+1
+2
+3
+4
+5
+6
+7
+8
+9

(h)


0
+1
+2
+3
+4
+5
+6
+7
+8
+9

3. (a) {all numbers}

(b) {all numbers greater than 1}

(c) {all numbers less than 1}

(d) The empty set. (The sentence has no solutions.)

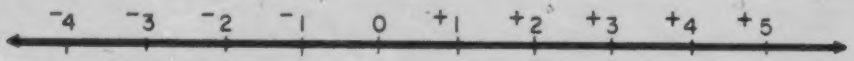
(e) {$^-5$}

(f) {$^-1$}

(g) {$^-1$}

(h) {$^-2$}

4. (a)


-4
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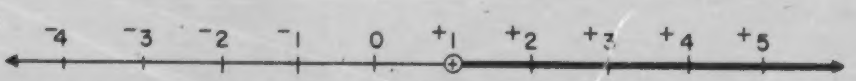
-1
0
+1
+2
+3
+4
+5

(b)


-4
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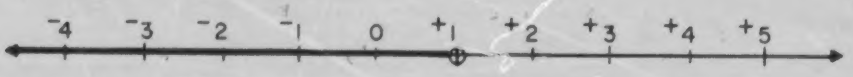
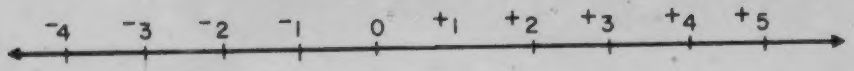
-2
-1
0
+1
+2
+3
+4
+5

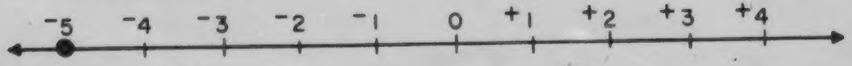
(c)


-4
-3
-2
-1
0
+1
+2
+3
+4
+5

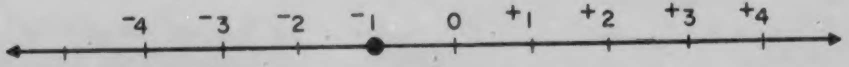
(d)


-4 -3 -2 -1 0 +1 +2 +3 +4 +5

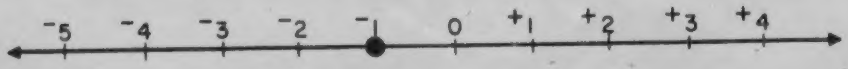
(e)


-5 -4 -3 -2 -1 0 +1 +2 +3 +4

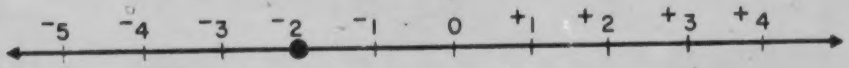
(f)



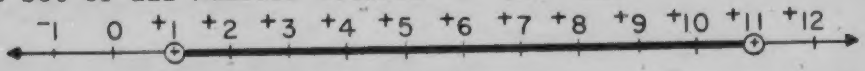
(g)


-5 -4 -3 -2 -1 0 +1 +2 +3 +4

(h)


-5 -4 -3 -2 -1 0 +1 +2 +3 +4

5. The set of all numbers between 1 and 11.


-1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10 +11 +12

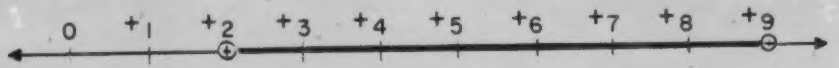
6. The intersection

7. (a) The set of all numbers less than 9 and greater than 2.

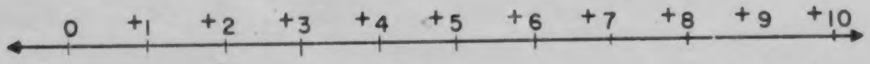
(b) The empty set.

(c) All numbers greater than 3 and less than 6.

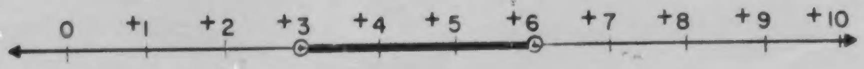
8. (a)


0 +1 +2 +3 +4 +5 +6 +7 +8 +9

(b)

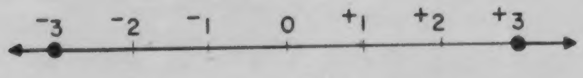

0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10

(c)


0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10

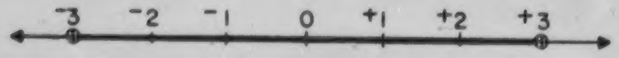
*9. (a) $\{3, {}^{-}3\}$

(b)

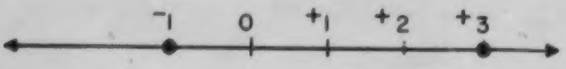


[pages 68-69]

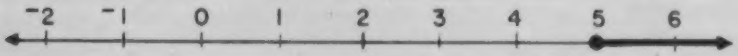
10. (a) All numbers greater than $^{-}3$ and less than 3.

(b) 

*11. (a) For $x + 7 = 6$ $x = {}^{-}1$
For $2x - 1 = 5$ $x = 3$
The set of solutions is $\{^{-}1, 3\}$

(b) 

*12. (a) For $x - 1 = 4$ $x = 5$
For $x - 1 > 4$ x is all numbers greater than 5.
The set of solutions is 5 and all numbers greater than 5.

(b) 

*13. The solution set of "$x < 10$" is all numbers less than 10.
The solution set of "$x - 9 > 0$" is all numbers greater than 9.
Every number is in one of these two sets (or both). Thus, the solution set of the compound sentence is the set of all numbers.

Answers to Class Exercises 2-2c

1. $80t = 560$
2. Let b stand for the number of years in the brother's age. Then $14 - 5 = b$, or $14 = 5 + b$.
3. Let s stand for the number of years in the sister's age. Then $10 + 4 = s$, or $10 = s - 4$.

4. Let k be the number of kits he bought. Then $25k = 75$.

5. Let b stand for the number of years in the boy's present age. Then $b + 7 = 20$.

6. Let f stand for the number of feet in the length of the board. Then $12f = 72$.

7. Let f stand for the number of feet in the length of the board. Then $f = 3 \cdot 5 = 15$, or $\frac{f}{3} = 5$.

8. Let a stand for the number of years in Ann's present age. Then $a - 10 = 3$.

9. Let d be the number of dollars. Then $d = \frac{450}{100}$.

10. Let m be the number of dollars Dick has. Then $2m + 3 < 23$.

11. Let s stand for the speed of the plane. $2s > 500$.

12. Let g stand for the number of years in the girl's age. Then $2g + 1 = 19$.

13. Let t stand for the number of hours the man drove. Then $40t = 240$.

14. Let c stand for the number of cents the baby sitter earns. Then $c = 5 \cdot 65$.

Answers to Exercises 2-2c

1. (a) $x + (x - 7) = 21$, that is, $2x - 7 = 21$.
   (b) $x = 2(x - 7)$
   (c) $3(x - 7) = x + 7$
   (d) $2x = 4(x - 7)$

2. (a) $m + 2m = 15$
   (b) $2m - 5 = m$
   (c) $2m = 5(m - 3)$

3. (a) $2w = (w + 4) + 3$
   (b) $2(w + 4) = 3w + 1$
   (c) $2w + 2(w + 4) = 36$

4. Let s be the number of years in Mr. Smith's present age. Then $s + 10 = 40$.

5. Let b stand for the number of dollars Bill has already earned. Then $b + 5 = 12$.

6. Let b stand for the number of inches in her brother's height. Then $2b = 64$.

7. Let y be the number of the year in which he was born. Then $y + 14 = 1958$.

8. Let n stand for the number wanted. Then $\frac{1}{5}n = 10$.

9. Let d stand for the number of inches in the length of the shorter piece. Then $d + (d + 10) = 50$.

10. Let v stand for the number of votes Bruce received. Then $b + (b + 5) = 35$.

11. Let n stand for the number. Then $n + 2n < 27$.

12. Let S stand for the number of people living in St. Paul. Then $1,000,000 > 2S$.

13. Let m stand for the number of pounds Mike weighs. Then $m \not> 105$.

14. Let a stand for the number of dollars in the boy's average monthly earnings. Then $12a > 120$.

15. Let x stand for the number of years in the son's age. Then $4x$ will represent the number of years in Mr. Smith's present age. The equation is $4x + 16 = 2(x + 16)$.

*16. Let x stand for the measure of the smallest angle. Then $x + 2x + 3x = 180$.

*17. Let n stand for the smallest of the three consecutive whole numbers. Then $n + (n + 1) + (n + 2) = 123$. (Other possibilities would be to let n stand for the middle number or the greatest number.)

*18. Let d stand for the number of dimes Bob has. Then $10d + 5(3d) = 125$.

[pages 74-75]

2-3. Finding Solution Sets

In this section we study some of the properties of the relation of equality and their applications to solutions of simple equations. The central idea is that of equivalent equations. If one is careful to consider in the process of solution only equations equivalent to the given one, then, unless an error is made, the final equation in the form $x = \text{some number}$, will show *the* solution. Only linear equations are considered since this is merely an introduction to the solution of equations. Inequalities are considered briefly in the next section.

You should notice three things:

1. No subtraction property or division property is needed. It would only complicate things to bring in such properties. Since $a - c = a + {}^{-}c$ and $\frac{a}{c} = a \cdot (\frac{1}{c})$ these properties are implied by the addition and multiplication properties.

2. The addition and multiplication properties themselves are not "axioms," nor do they add anything *new* to our fund of information about numbers, equality, and the operations. For instance, we already know that addition is associative, that $c + {}^{-}c = 0$, that $b + 0 = b$, and that subtraction is the inverse operation for addition. Thus, if, $a = b$ then $a = b + 0$ or $a = b + (c + {}^{-}c)$. By the associative property, $a = (b + c) + {}^{-}c$ or $a = (b + c) - c$. By the very definition of subtraction this last statement is equivalent to $a + c = b + c$, and we have proved what is called the "addition property" in the text. You may wish to point this out to your class.

3. We could perfectly well solve the simple equations in this section without any mention of the addition and multiplication properties. For example, $z + 3 = {}^{-}7$ is equivalent to $z = {}^{-}7 - 3 = {}^{-}10$ by the definition of subtraction. We have chosen to present the material as it is in the text for two reasons: (a) These properties are important and useful in themselves; working with them gives excellent opportunity for review of the associative, commutative, and even the distributive properties of the operations. (b) We wish to avoid planting the seed of "transfer-from-one-side-to-the-other-and-change-the-sign."

[page 75]

Exercises 2-3a are still somewhat exploratory and review certain ideas which will be needed immediately ahead. There is some similarity between Class Exercises 2-3c and 2-3d. The purpose is to have the students try some of the more difficult equations in the former and then in Class Exercises 2-3d to explore a little more of what is involved before attempting equations again.

Answers to Class Exercises 2-3a

1. $x = 3$, $x + 2 = 5$, $x + 5 = 8$, $x + {}^-5 = {}^-2$, etc. Many answers are possible.
2. $x + 1 = 4$, $x + 2 = 5$, etc. All the answers to Problem 1 are answers to this problem as well.
3. $2x + 3 = 15$, $2x + 5 = 17$, $2x + {}^-3 = 9$, $x = 6$, $x + 1 = 7$, etc.
4. One may add the same number to both sides of the equation, or multiply both sides by the same non-zero number.
5. If Equation E has the same set of solutions as Equation F, then F must have the same set of solutions as E.

*6. Yes they are equivalent, because the solution set of each is the null set.

Answers to Class Exercises 2-3b

1. (a) 6. Check: $6 + 4 = 10$
   (b) 7. Check: $7 + {}^-2 = 5$
   (c) $^-1$. Check: $3 = {}^-1 + 4$
   (d) 1. Check: ${}^-2 = 1 + {}^-3$
2. (a) $5x$
   (b) $6x$
   (c) x
   (d) $6x$
   (e) $({}^-3)x$
   (f) $({}^-7)x$

[pages 75-78]

Answers to Exercises 2-3a

1. (a) $x = 6$, $x + 3 = 9$, $2x + 14 = 26$, $x + {}^-3 = 3$

   (b) $x = 20$, $x + 1 = 21$, $23 = x + 3$, $19 = x - 1$

   (c) $x + 1 = 8$, $2x = 14$, $x + 16 = 23$, $x + {}^-2 = 5$.

   Many answers are possible for each part.

2. (a)

   $x + 5 = 6$

   $(x + 5) + {}^-5 = 6 + {}^-5$ addition property, adding $^-5$.

   $x + (5 + {}^-5) = 1$ associative property

   $x = 1$ $5 + {}^-5 = 0$

   If $x = 1$, then $x + 5 = 1 + 5 = 6$, so 1 is a solution.

   (b)

   $x + 6 = 5$

   $(x + 6) + {}^-6 = 5 + {}^-6$ addition property, adding $^-6$.

   $x + (6 + {}^-6) = {}^-1$ associative property

   $x = {}^-1$ $6 + {}^-6 = 0$

   If $x = {}^-1$, then $x + 6 = {}^-1 + 6 = 5$, so $^-1$ is a solution.

   (c)

   $x + {}^-7 = 7$

   $(x + {}^-7) + 7 = 7 + 7$ addition property, adding 7.

   $x + ({}^-7 + 7) = 14$ associative property

   $x = 14$ ${}^-7 + 7 = 0$

   If $x = 14$, then $x + {}^-7 = 14 + {}^-7 = 7$, so 14 is a solution.

   (d) 0. Check: $0 - 7 = {}^-7$

   (e) $^-19$. Check: ${}^-19 + 6 = {}^-13$

   (f) 1. Check: $4 = 1 + 3$

   (g) 2. Check: ${}^-2 = {}^-4 + 2$

   (h) $\frac{17}{2}$. Check: $\frac{17}{2} + \frac{3}{2} = 10$

(i) 4. Check: $4 - \frac{3}{2} = \frac{5}{2}$

(j) $^-(\frac{61}{5})$. Check: $^-(\frac{61}{5}) + 14 = \frac{9}{5}$

(k) $\frac{6}{7}$. Check: $\frac{13}{7} = 1 + \frac{6}{7}$

(l) $^-(\frac{11}{117})$. Check: $^-(\frac{11}{117}) + {}^-(\frac{4}{9}) = {}^-(\frac{7}{13})$

3. The resulting equation is:

(a) $2x = 5$

(b) $3x = {}^-5$

(c) $5x = {}^-4$

(d) $10x = 10$

(e) $2u = 10$

(f) $2x = 12$

(g) $^-4y = 3 + \frac{9}{2} = \frac{15}{2}$

4. (a) $^-3$

(b) 7

(c) The numbers 3 and $^-3$ are additive inverses of each other.

(d) The numbers 7 and $^-7$ are additive inverses of each other.

5. (a) $^-x = {}^-3$

(b) $^-x = {}^-(\frac{1}{2})$

(d) $x = \frac{1}{2}$

(e) $x = 7$

6. If we add $^-(4x)$ to both sides of the equation $4x + 5 = 3x + 2$, we can carry through the solution as follows (this is not the only way):

$$\begin{aligned} {}^-(4x) + 4x + 5 &= {}^-(4x) + 3x + 2 \\ 5 &= {}^-x + 2 \\ 3 &= {}^-x. \end{aligned}$$

Then, since $^-(^-a) = a$, we have $^-3 = x$.

7. (a) $2x$

(b) $2x$

(c) x

8. Add ^{-}x to both sides to get $x + 7 = 0$. Add $^{-}7$ to both sides to get $x = {}^{-}7$.

9. Add ^{-}x to both sides to get $x + 3 = 2$. Add $^{-}3$ to both sides to get $x = {}^{-}1$.

10. Add ^{-}x to both sides to get $0 = x + 6$. Add $^{-}6$ to both sides to get $x = {}^{-}6$.

11. Add $^{-}(2x)$ to both sides to get $x + 5 = 0$. Add $^{-}5$ to both sides to get $x = {}^{-}5$.

Answers to Class Exercises 2-3c

1. (a) Addition property. Add $^{-}10$.
   (b) Addition property. Add $^{-}6.2$.
   (c) Addition property. Add 2.
   (d) Multiplication property. Multiply by $\frac{1}{5}$.
   (e) Multiplication property. Multiply by 18.
   (f) Addition property. Add x.
   (g) Multiplication property. Multiply by 2.
   (h) Addition property. Add $^{-}18$.
   (i) Addition property. Add $^{-}6$.
   (j) Multiplication property. Multiply by $\frac{1}{0.08}$.
   (k) Addition property. Add $y + {}^{-}19$. This could also be accomplished by first adding y to both sides and then $^{-}19$.
   (l) Multiplication property. Multiply by $\frac{3}{2}$.
   (m) Multiplication property. Multiply by $\frac{1}{45}$.
   (n) Multiplication property. Multiply by c.

2. (b) $\frac{1}{5}$
   (c) $^{-}(\frac{1}{3})$
   (d) 2
   (e) 4
   (f) $\frac{3}{2}$
   (g) $^{-}(\frac{2}{1})$

3. (a) Additive inverse: $^{-}3$ Multiplicative inverse: $\frac{1}{3}$
   (b) Additive inverse: $-\frac{1}{2}$ Multiplicative inverse: 2
   (c) Additive inverse: $\frac{^{-}5}{7}$ Multiplicative inverse: $\frac{7}{5}$

Answers to Exercises 2-3b

1. (a) Additive inverse: $^{-}7$ Multiplicative inverse: $\frac{1}{7}$
   (b) Additive inverse: $^{-}(\frac{3}{4})$ Multiplicative inverse: $\frac{4}{3}$
   (c) Additive inverse: 2 Multiplicative inverse: $^{-}(\frac{1}{2})$
   (d) Additive inverse: $^{-}(\frac{1}{2})$ Multiplicative inverse: 2

2. (a) Add $^{-}2$ to get $3x = 12$. Multiply by $\frac{1}{3}$ to get $x = 4$.
   (b) Multiply both sides by $\frac{1}{7}$ to get $x = \frac{2}{7}$.
   (c) Add $3x$ to both sides to get $7 = 22 + 3x$. Then add $^{-}22$ to both sides to get $^{-}15 = 3x$. Multiply both sides by $\frac{1}{3}$ to get $^{-}5 = x$.
   (d) Multiply both sides by 2 to get $x = 14$.
   (e) Multiply both sides by $^{-}2$ to get $x = {}^{-}28$.

Answers to Class Exercises 2-3d

1. (a) Addition property, adding $^{-}1$.
   (b) Multiplication property, multiplying by $\frac{1}{4}$.

(c) Multiplication property, multiplying by 3.

(d) Multiplication property, multiplying by $^{-}1$.

(e) Multiplication property, multiplying by $\frac{1}{2}$.

(f) Multiplication property, multiplying by $\frac{1}{2}$.

(g) Multiplication property, multiplying by 10.

(h) Multiplication property, multiplying by n.

(i) Addition property, adding $^{-}(3x)$.

2. (a) Add 2 to both sides to get $3y = 9$, multiply both sides by $\frac{1}{3}$ to get $y = 3$. Check: $3 \cdot 3 + {}^{-}2 = 7$.

(b) Add $^{-}1$ to both sides to get $6 = 3x$, multiply both sides by $\frac{1}{3}$ to get $2 = x$. Check: $7 = 3 \cdot 2 + 1$

(c) Multiply both sides by $\frac{1}{3}$ to get $2 = w$. Check: $6 = 3 \cdot 2$.

(d) Add 1.7 to both sides to get $\frac{1}{2}t = 0.4$, multiply both sides by 2 to get $t = 0.8$. Check: $\frac{1}{2}(0.8) - 1.7 = 0.4 - 1.7 = {}^{-}1.3$.

(e) Multiply both sides by 18 to get $36 = x$. Check: $2 = \frac{36}{18}$.

(f) Add $^{-}(0.14)$ to both sides to get $x = 5.14$. Check: $0.14 + 5.14 = 5.28$.

(g) Add 7 to both sides to get $5x = 2x + 7$, add $^{-}(2x)$ to both sides to get $3x = 7$, multiply both sides by $\frac{1}{3}$ to get $x = \frac{7}{3}$. Check: $5 \cdot \frac{7}{3} = 2 \cdot \frac{7}{3} + 7 = \frac{14 + 21}{3}$.

(h) Add $2x$ to both sides to get $3x = 7$. Multiply both sides by $\frac{1}{3}$ to get $x = \frac{7}{3}$. Check: $5 \cdot \frac{7}{3} - 7 = 2 \cdot \frac{7}{3}$.

Answers to Exercises 2-3c

1. (a) $x = 3$
   (b) $y = 8$
   (c) $t = {}^{-}2$
   (d) $x = \frac{1}{3}$

2. (a) $x = 2$
   (b) $y = {}^{-}8$
   (c) $v = 1$
   (d) $m = {}^{-}4$
   (e) $y = 8$
   (f) $u = 8$
   (g) $w = 4$
   (h) $s = 4$
   (i) $w = 2$
   (j) $w = {}^{-}2$
   (k) $t = {}^{-}4$
   (l) $2 = w$

3. (a) $\frac{1}{9}$
   (b) 3
   (c) $\frac{5}{4}$
   (d) They are inverses, relative to multiplication
   (e) They are inverses, relative to multiplication
   (f) They are inverses, relative to multiplication

4. (a) Multiply by $\frac{1}{4}$ to get $x + 1 = 3$ and then add ${}^{-}1$ to get $x = 2$.

   (b) Multiply both sides by $\frac{1}{7}$ to get $x - 2 = \frac{13}{7}$, add 2 to both sides to get $x = \frac{27}{7}$.

   (c) Multiply both sides by 3 to get $x + 9 = 15$ and then add ${}^{-}9$ to both sides to get $x = 6$.

   (d) Multiply both sides by $\frac{1}{0.6}$ to get $x - 0.3 = \frac{1}{3}$. Add 0.3 to both sides to get $x = 0.3 + \frac{1}{3} = \frac{19}{30}$.

   (e) Multiply by 2 to get $3x + 4 = 14$, add ${}^{-}4$ to both sides to get $3x = 10$, multiply both sides by $\frac{1}{3}$ to get $x = \frac{10}{3}$. (In this case, any multiple of 2 may be used as a multiplier.)

(f) Multiply both sides by 0.12 to get $4x + 1 = 0.36$. Add $^{-}1$ to both sides to get $4x = {}^{-}0.64$, multiply both sides by $\frac{1}{4}$ to get $x = {}^{-}0.16$.

5. Add x to both sides of the equation $^{-}x + 4 = 1 + 4x$. This gives $4 = 1 + 5x$. Then add $^{-}1$ to both sides to get $3 = 5x$. Multiplication by $\frac{1}{5}$ gives $\frac{3}{5} = x$.

2-4. Solving Inequalities

It is the purpose of this section to deal only briefly with the solution of inequalities. The development emphasizes the parallelism between the addition property for equality and that for inequality. Since there are difficulties with the multiplication property, this is avoided except in an exercise or two. Here the chief purpose is to show that the methods already learned apply to a *limited* extent in inequalities as well.

The difficulty with the multiplication property involving inequalities is that while the property holds true for positive numbers, it does not hold true for negative numbers.

For addition, if x, y, and z are numbers and $x < y$, then $x + z < y + z$. This is also true for $x > y$.

For multiplication, if x, y, and z are numbers and $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$. Notice that this property differs slightly from the multiplication property of "=".

Note, however, that if x, y, and z are numbers and $x < y$ and $\underline{z < 0}$ (that is, z is a negative number), then $x \cdot z > y \cdot z$. This "reversing" of the last inequality is very confusing. Teachers are urged to use the number line to satisfy themselves that this reversal holds true.

Answers to Exercises 2-4

1. (a) Add $^-5$ to both sides to get $x < 2$.

   (b) Add $^-5$ to both sides to get $2 > x$.

   (c) Add 2 to both sides to get $x < 10$.

   (d) Add 3 to both sides to get $y < 13$.

   (e) Add 3 to both sides to get $13 < y$.

   (f) Add ^-x to both sides to get $x + 3 < 2$ and then $^-3$ to both sides to get $x < {}^-1$.

   (g) Add ^-x to both sides to get $4 < 5$. This is true for all values of x, and, hence, the given inequality is true for all values of x as well.

   (h) Add ^-x to both sides to get $4 > 5$ which is false for all values of x, and, hence, the given inequality is false for all values of x.

   (i) Add $^-(2x)$ to both sides to get $x + 2 < {}^-3$. Then add $^-2$ to both sides to get $x < {}^-5$.

   (j) Add $\frac{1}{2}x$ to both sides to get $x + 3 > 4$ and then add $^-3$ to both sides to get $x > 1$.

   (k) Add ^-x to both sides to get $7 + x < {}^-(\frac{1}{7})$. Then add $^-7$ to both sides to get $x < {}^-(\frac{50}{7})$.

2. If c is a negative number, then the point which $a + c$ represents will be on the number line, the same number of units to the left of the point which a represents as $b + c$ is to the left of the point which b represents. Since $a < b$, the point represented by $a + c$ will therefore be to the left of the point represented by $b + c$, and, hence,
$$a + c < b + c.$$

3. We have shown that if $a < b$, then $a + c < b + c$. If we replace a by $(a + c)$, b by $(b + c)$, c by ^{-}c we see that
if $(a + c) < (b + c)$, then $(a + c) + {}^{-}c < (b + c) + {}^{-}c$.
Thus:
$$a + (c + {}^{-}c) < b + (c + {}^{-}c),$$
that is,
$$a < b,$$
which is the conclusion we wanted.

4. If $a > b$ then the point which a represents will be to the right of the point which b represents. The point which $a + c$ represents will be the same distance from a and in the same direction as $b + c$ is from b. Hence, the point which $a + c$ represents must be to the right of the point which $b + c$ represents and
$$a + c > b + c.$$

5. An indirect proof may be used here as follows: Suppose $a + c = b + c$. Then it would follow that $a = b$, which contradicts our assumption.

6. One way to show this is by the following sequence of steps: $2a = a + a < a + b$ by the addition property. Again, by the addition property, $a + b < b + b = 2b$. So the point representing $2a$ is to the left of the point representing $a + b$ which, in turn, is to the left of the point representing $2b$. Hence, the point representing $2a$ is to the left of the point representing $2b$.

*7. If $a < b$ it may be seen from the number line that $^{-}a > {}^{-}b$. Then $^{-}(2a) = {}^{-}a + {}^{-}a > {}^{-}a + {}^{-}b > {}^{-}b + {}^{-}b = {}^{-}(2b)$.

2-5. Number Sentences with Two Unknowns

This section should be lightly touched on. No difficult graphs should be attempted. This is mere exploration so that it can be seen how simple graphs may be made and what relationships they have to the solutions of equations.

This section introduces the important concept of an equation or inequality involving two unknowns. Here the solution is an ordered pair of numbers. This leads in turn to the representation of the solution of a number sentence by some type of graph.

It is especially important that the students work through this section with pencil and paper. All of them should have graph paper before they begin reading this part. You may pattern the class discussion on the questions in the text.

In studying ordered pairs of numbers, the teacher should emphasize the importance of order in performing operations, such as subtractions and divisions as well as in many non-mathematical situations.

In the class discussions of $y = x^2$, the students should notice that for each positive value of y there are two values for x, and that no value of x will give a negative value for y. It may be worthwhile making a magnified graph in the neighborhood of the origin taking x as 0, ± 0.1, ± 0.2, etc.

In plotting straight lines it might be mentioned that locating two points is sufficient, but that as a check it is best to plot at least three points--preferably more.

There is one exception to the general statement that the graph of $ax + by = c$ is a line. There is no need to discuss this topic in class, but the teacher should be prepared in case any bright student notices this possibility.

The exceptional case occurs when both a and b are zero. If c is also zero, the equation becomes $0x + 0y = 0$. This equation is satisfied by all values of x and y, and therefore the graph is the entire plane.

If, however, c is not zero, then the equation (for $c = 5$) becomes $0x + 0y = 5$. This equation is satisfied by no values of x and y, and the graph is the empty set.

The class will probably agree that whenever we refer to $ax + by = c$ as a linear equation, we are implying that at least one of the terms in x and y actually appears with a non-zero coefficient.

Answers to Questions in Section 2-5

The sentence: $x + 1 = y$.

If $x = 3$ and $y = 5$ the sentence is false.

If $x = 7$, y must be 8.

If $y = {}^{-}6$, x must be ${}^{-}7$.

The table is:

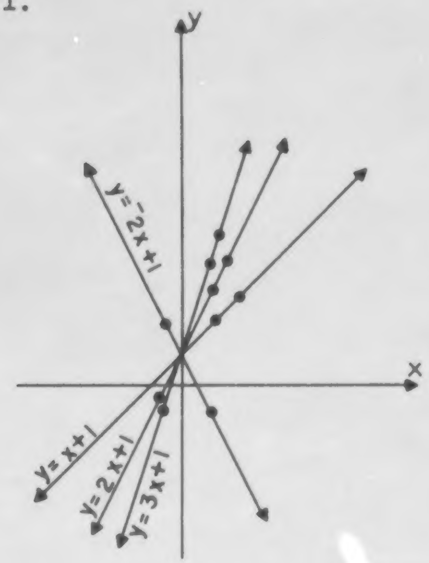
x	y
0	1
1	2
2	3
${}^{-}(\frac{2}{3})$	$\frac{1}{3}$
${}^{-}(\frac{16}{3})$	${}^{-}(\frac{13}{3})$

The equation $2x + y = {}^{-}1$.

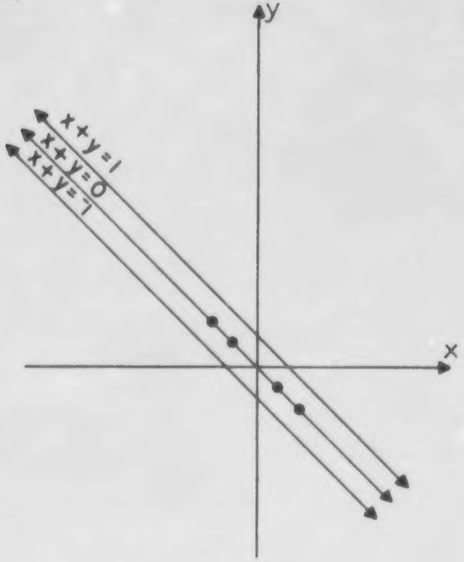
x	y
${}^{-}3$	5
${}^{-}1$	1
0	${}^{-}1$
4	${}^{-}9$
$\frac{1}{2}$	${}^{-}2$
${}^{-}(\frac{1}{2})$	0
${}^{-}(\frac{3}{2})$	2

Answers to Exercises 2-5a

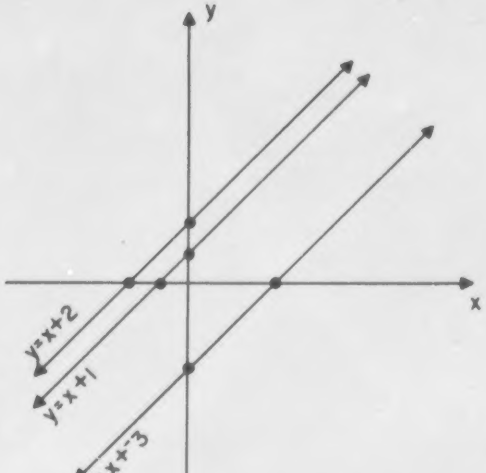
1.



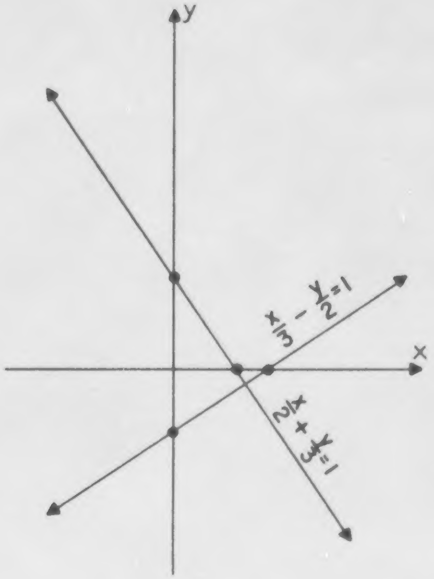
3.



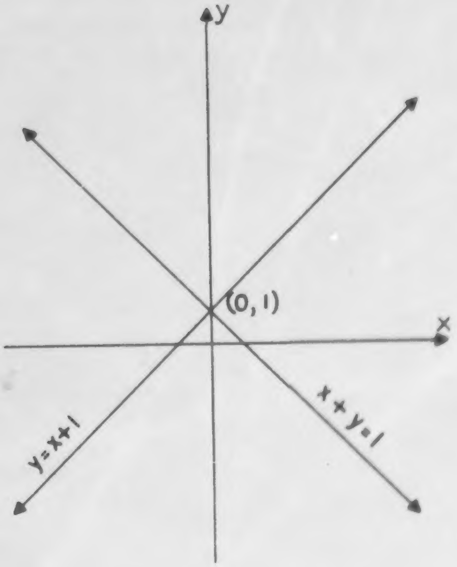
2.



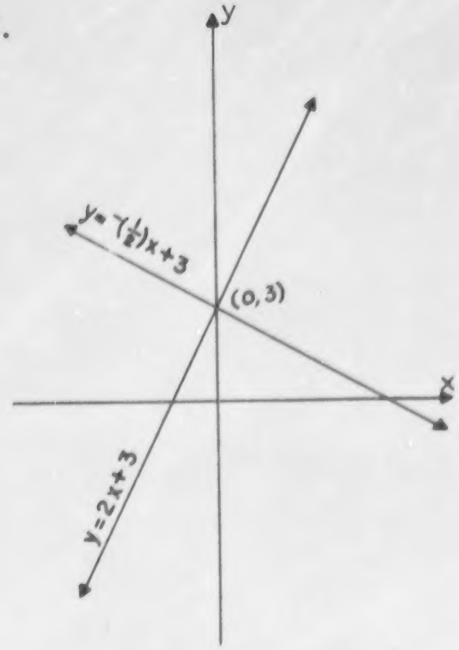
4.



5.



6.



Answers to Questions in Section 2-5

The number sentence "$2x + 2y = 16$ and $x > 0$, $y > 0$." The pairs (1, 7), (2, 6), $(\frac{7}{2}, \frac{9}{2})$, and (5, 3) are solutions. The solutions (1, 7), (2, 6), and (5, 3) have been plotted on the graph.

The inequality $xy > 0$.

The blanks should contain positive, negative, first, third.

The equation $y = x^2$

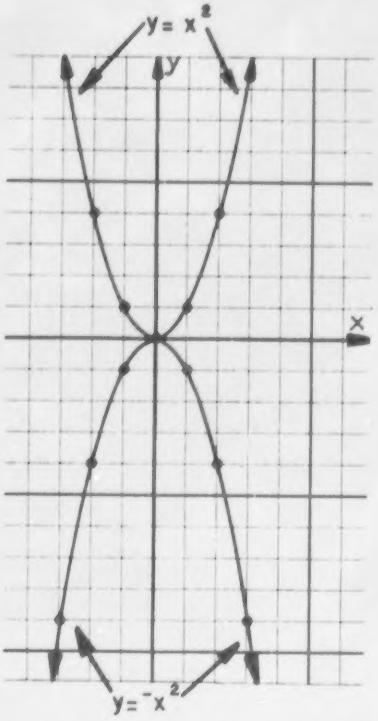
x	y
$^{-}4$	16
$^{-}3$	9
$^{-}2$	4
$^{-}1$	1
0	0

x	y
1	1
2	4
3	9
4	16

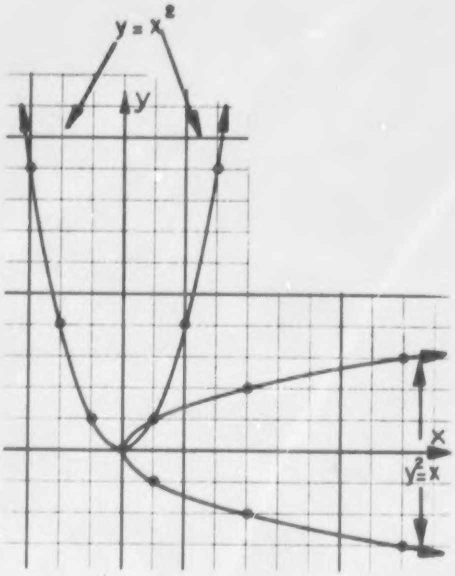
If $y > x^2$, the point with coordinates (x, y) will lie above the point (x, x^2). If a point (x, y) lies above the parabola, then $y > x^2$.

Answers to Exercises 2-5b

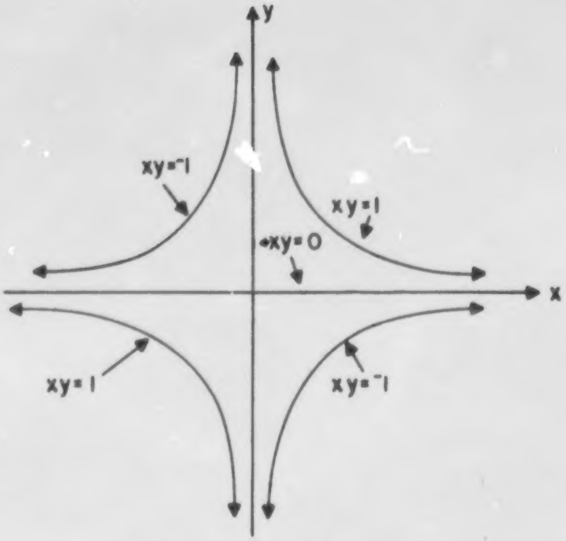
1. (a)



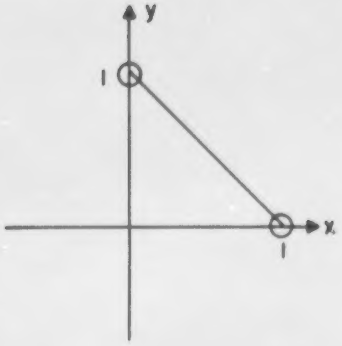
(b)



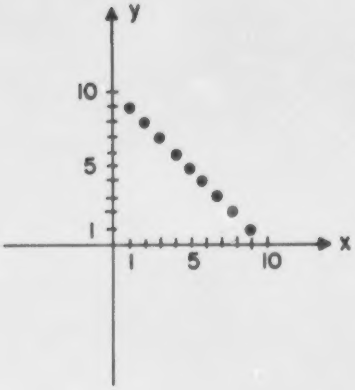
(c) $xy = 0$ is the union of the set of points on the x-axis and the set of points on the y-axis.


y
xy=⁻1
xy=1
xy=0
x
xy=1
xy=⁻1

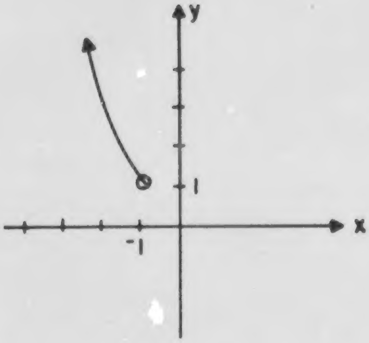
2. (a)



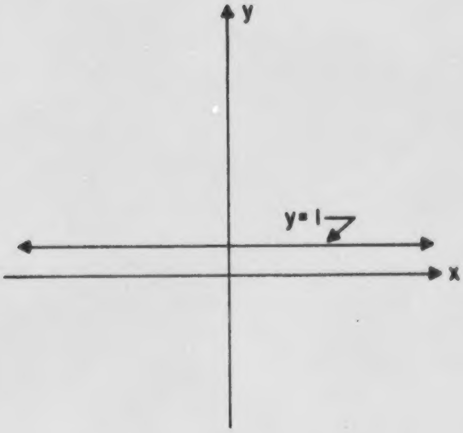
(b)


y
10
5
1
1
5
10
x

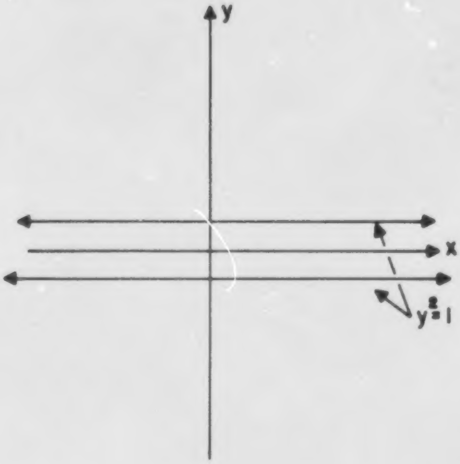
(c)



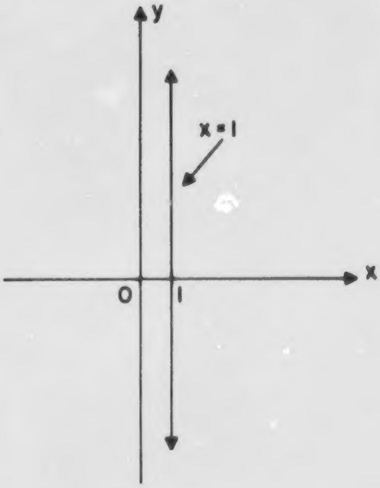
(d)



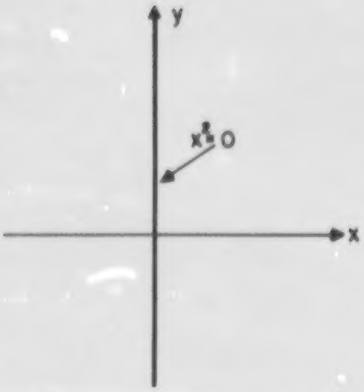
(e)



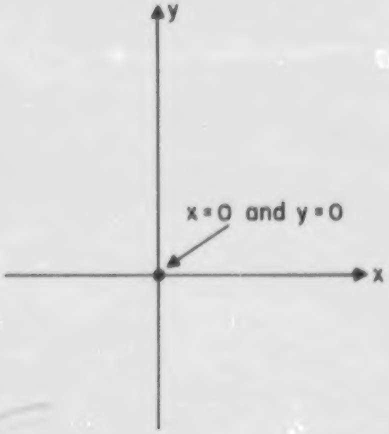
(f)



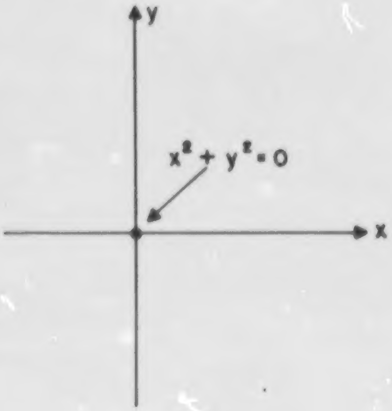
(g)



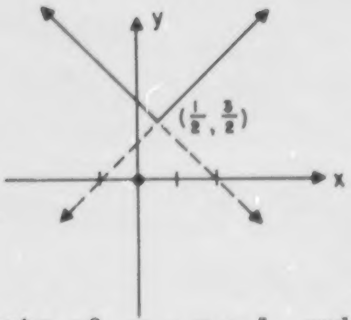
(h)



(i)

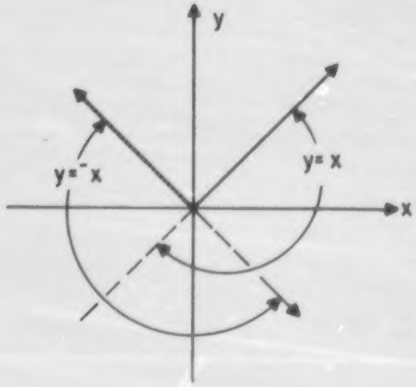


*(j)


y
$(\frac{1}{2}, \frac{3}{2})$
x

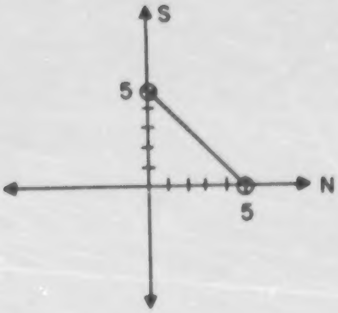
Graphs of $y = x + 1$ and $y = 2 - x$ shown. Graph of the number sentence in 2(j) is the union of the two darkened rays.

*(k) The graph of the number sentence is the union of the two darkened rays.



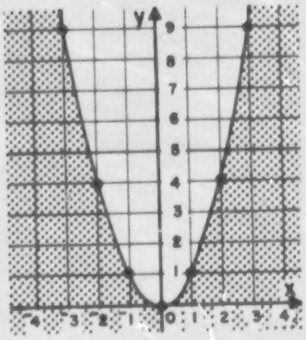
3.	(a)	(0, 1), (1, 0)	2 solutions
	(b)	(0, 2), (1, 1), (2, 0)	3 solutions
	(c)	(0, 20), (1, 19), (2, 18), (3, 17), (4, 16), (5, 15), (6, 14), (7, 13), (8, 12), (9, 11), (10, 10), (11, 9), (12, 8), (13, 7), (14, 6), (15, 5), (16, 4), (17, 3), (18, 2), (19, 1), (20, 0)	21 solutions
	(d)	(0, 0)	1 solution
	(e)	(1, 0)	1 solution
	(f)	(2, 0), (0, 1)	2 solutions
	(g)	(3, 0), (1, 1)	2 solutions
	(h)	(4, 0), (2, 1), (0, 2)	3 solutions
	(i)	(1, 12), (3, 11), (5, 10), (7, 9), (9, 8), (11, 7), (13, 6), (15, 5), (17, 4), (19, 3), (21, 2), (23, 1), (25, 0)	13 solutions
	(j)	(0, 5), (7, 0)	2 solutions
	(k)	(3, 3)	1 solution
	(l)	(6, 1)	1 solution

4.



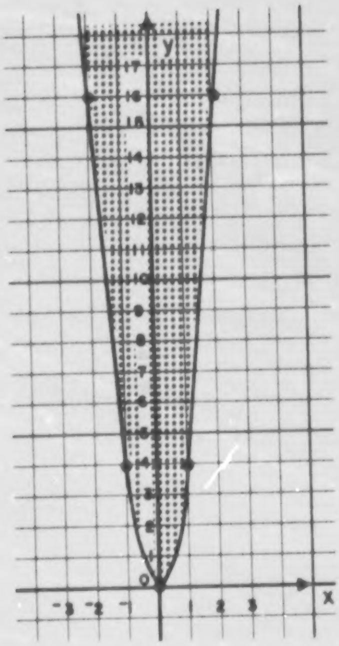
$s + n = 5$ and $s > 0$, $n > 0$.

5. (a)



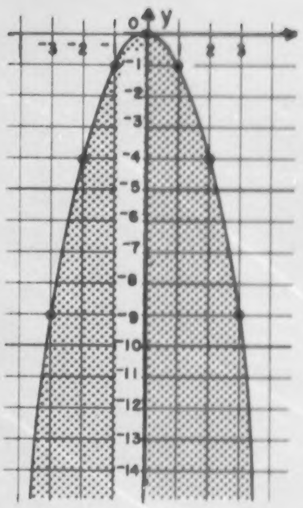
$y < x^2$

(b)



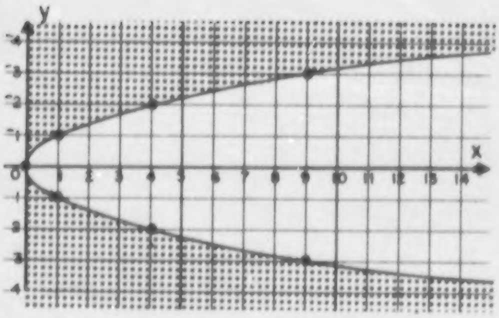
$y > 4x^2$

(c)


y
o
-3
-2
-1
2
3
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
-13
-14

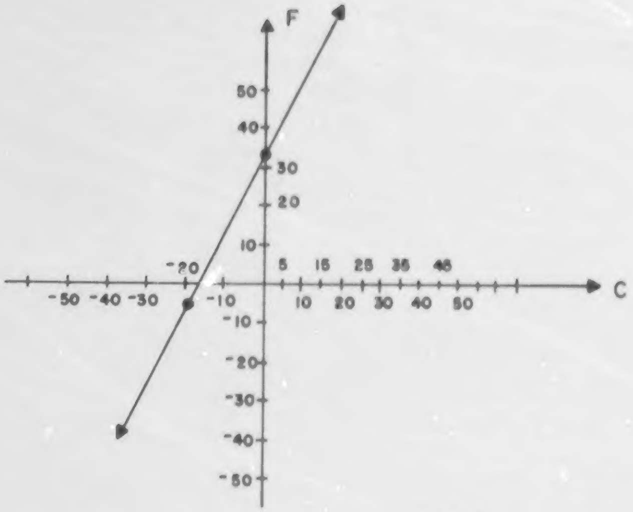
$y < -(x^2)$

(d)


y
x
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
4
3
2
1
-1
-2
-3
-4

$y^2 > x$

6.


F
C
50
40
30
20
10
-10
-20
-30
-40
-50
-20
5
15
25
35
45
-50
-40
-30
-10
10
20
30
40
50

(a) If $C = {}^{-}25$ then $F = {}^{-}13$

If $C = {}^{-}15$ then $F = 5$

If $C = 0$ then $F = 32$

If $C = 4$ then $F = \frac{196}{5} = 39\frac{1}{5}$

(b) If $F = {}^{-}30$ then $C = {}^{-}(\frac{310}{9}) = {}^{-}34\frac{4}{9}$

If $F = {}^{-}15$ then $C = {}^{-}(\frac{235}{9}) = {}^{-}26\frac{1}{9}$

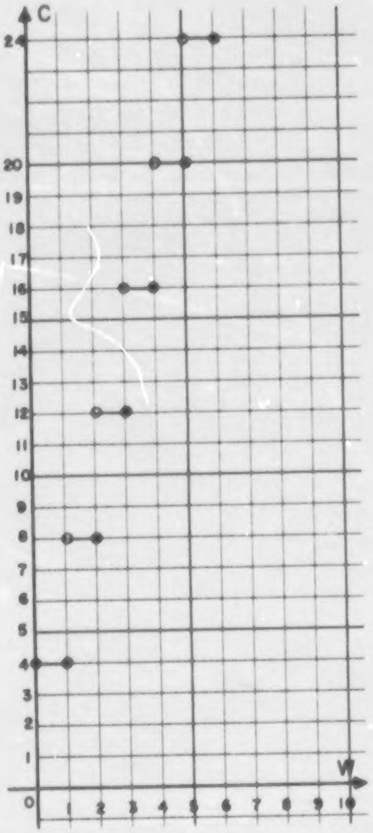
If $F = 0$ then $C = {}^{-}(\frac{160}{9}) = {}^{-}17\frac{7}{9}$

If $F = 50$ then $C = 10$

*7.

w	c
$w > 0$ and $w \leq 1$	4
$1 < w \leq 2$	8
$2 < w \leq 3$	12
$3 < w \leq 4$	16
$4 < w \leq 5$	20
$5 < w \leq 6$	24

Note: In the graph the circle means that number is not included. The large dot means the number is included.



8. The temperature is $^-40$ degrees. Edward had said that $F = C$, and we know that $F = \frac{9}{5}C + 32$. Thus, $C = \frac{9}{5}C + 32$. Adding $^-(\frac{9}{5})C$, using the addition property, we have

$$C + {}^-(\tfrac{9}{5})C = {}^-(\tfrac{9}{5})C + (\tfrac{9}{5}C + 32)$$

or

$$\left(1 + {}^-(\tfrac{9}{5})\right)C = 32$$

or

$$^-(\tfrac{4}{5})C = 32.$$

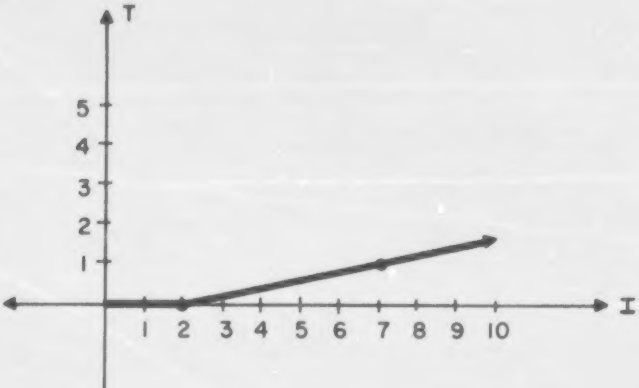
Multiplying by $^-(\frac{5}{4})$ using the multiplication property, we have

$$C = {}^-(\tfrac{5}{4}) \cdot 32 = \frac{^-5 \cdot 32}{4} = \frac{^-5 \cdot 8 \cdot 4}{1 \cdot 4} = \frac{^-5 \cdot 8}{1} = {}^-40.$$

Checking to see whether $^-40$ is a solution: If $C = {}^-40$ then $\frac{9}{5}C + 32 = \frac{9}{5}(^-40) + 32 = 9(^-8) + 32 = {}^-72 + 32 = {}^-40$. Thus, $C = {}^-40 = \frac{9}{5}C + 32$, so $^-40$ is a solution.

9. (a) T = the larger of the numbers $(.2)I - .4$ and 0, and $I \geq 0$.

(b)



(Each unit on the I and T axis represents $1,000.)

(c) If $I = 10$ then $T = 1.6$, so the tax on an income of \$10,000 is \$1600.
If $I = 3.5$ then $T = .3$, so the tax on an income of \$3500 is \$300.
If $I = 1.5$ then $T = 0$, so the tax on an income of \$1500 is \$0.
If $T = 1.5$, then $I = 9.5$, so that his income is \$9500.

Answers to Review Exercises

1. $x + (x + 3) = 45$
shorter piece is 21 inches long
longer piece is 24 inches long.

2. If x represents the width, then

$$2x + 2(x + 10) = 68$$
$$x = 12 \quad \text{(width)}$$
$$x + 10 = 22 \quad \text{(length)}$$

If x represents the length, then

$$2(x - 10) + 2x = 68$$
$$x = 22 \quad \text{(length)}$$
$$x - 10 = 12 \quad \text{(width)}.$$

3. $\$6000 + x + 2x = \$10,500$

$$x = \$\ 1,500 \quad \text{(son's share)}$$
$$2x = \$\ 3,000 \quad \text{(daughter's share)}.$$

4. $x + 15 < 51$

$$\{1, 2, 3, 4, \ldots, 35\}$$

5. $500 - x < 100$

$$\{401, 402, \ldots, 499\}.$$

(Note: It should be assumed he has at least one kit left, and he cannot sell more than 500.)

6. $x \neq 176$

$\{0, 1, \ldots, 176, 178, 179, \ldots\}$.

(Note: The limits were not established, but obviously there should be some limitation.)

7. $3x - x = 12$

$x = 6$

8. $x + 5x + 10x + 25x + 50x = 273$

$x = 3$

9. $x + (x + 30) < 316$

$x < 143$

Don received fewer than 143 votes.

10. $(x - 5) + (x - 5) = 18$

or

$2(x - 5) = 18$

$x = 14$

11. $x + 5x + 2(5x) = 48$

$x = 3$ ($5 bills)

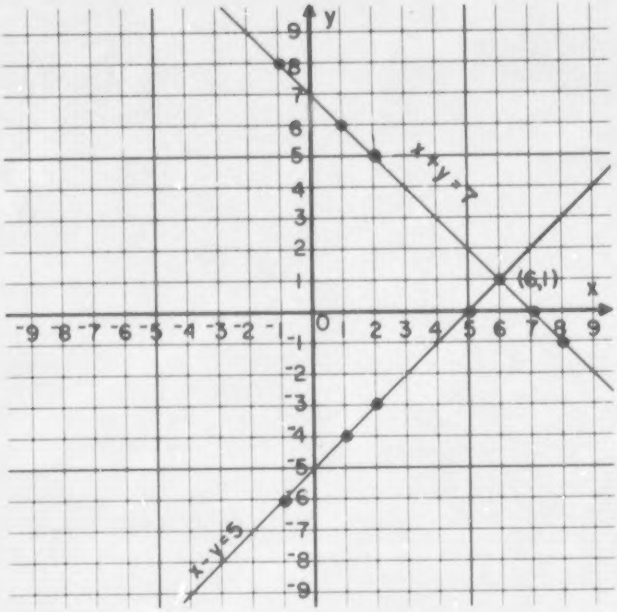
12. (a) $x + y = 7$ (b) $x - y = 5$

13. (a) $x + y = 7$ (b) $x - y = 5$

x	y
1	6
2	5
0	7
$^{-}1$	8
8	$^{-}1$

x	y
1	$^{-}4$
2	$^{-}3$
0	$^{-}5$
$^{-}1$	$^{-}6$
4	$^{-}1$

(c)



(d) (6, 1)

14. (a) $x = 6$, $y = 1$

 (b) This is the only pair since two distinct lines can have only one point in common.

*15. (0, 0) since $x = y$ and $x = {}^-y$ means that $y = {}^-y$, $2y = 0$ and $y = 0$.

*16. Here $x + 1 = x - 1$. This is not possible, and, hence, the set of solutions is the null set. The two lines are parallel.

*17. Let x stand for the first of the three consecutive integers. Then the third is $x + 2$ and the condition is: $x + (x + 2) = 192$. Then $2x + 2 = 192$, $x + 1 = 96$ and $x = 95$. Hence, the three consecutive integers are: 95, 96, 97 and $95 + 97 = 192$.

*18. Let x stand for the number of degrees in the measure of the two congruent angles. Then $2x$ is the measure of the third and we have $2x + x + x = 180$. This gives $4x = 180$, or $x = 45$.

*19. Let t be the number of hours they travel before being 750 miles apart. The Jones family will travel 55t miles and the Smith family 45t. Then we have the equation $55t + 45t = 750$. This becomes $100t = 750$ and $t = 7\frac{1}{2}$. Hence, they travel $7\frac{1}{2}$ hours.

*20. Let t stand for the number of hours it will take one of the carrier's planes to intercept the contact. The distance covered by the contact before the interception is 350t miles. The distance covered by the plane from the carrier is 450t. The equation then is $350t + 450t = 400$. This becomes $800t = 400$, $t = \frac{1}{2}$. Hence, interception will take place in half an hour.

*21. Here, using the expressions above, we have the equation

$$10 + 350t = 450t.$$

This equation is equivalent to $10 = 100t$, and, hence, $t = \frac{1}{10}$. Thus, it will take one-tenth of an hour, or six minutes to intercept the contact. In $\frac{1}{10}$ of an hour the carrier plane will travel $\frac{450}{10} = 45$ miles.

Sample Questions

True-False

T 1. If two numbers are unequal, then one must be greater than the other.

F 2. "Open sentence" is another name for a "mathematical phrase."

F 3. The solution set of an open sentence must contain at least one element. (Note: The idea of an empty or vacuous set is not discussed in the text for pupils but is touched upon in the Teachers' Commentary.)

T 4. $4(5 - 2)$ and $\frac{(6 \cdot 2)}{1}$ are numerals for the same number.

F 5. $3(\frac{2}{3} + 5) < 17$.

F 6. $\frac{4}{2} \cdot 0 = 8(\frac{1}{4})$.

T 7. A phrase is a part of a sentence.

T 8. The expression $x + 5$ is called an open phrase.

T 9. The expression "$(6a + 2b + 3c)$" represents or names one number.

F 10. The addition property tells us that equals may be added to equals.

T 11. "$4 + 2 = 7$" is an equation.

T 12. "=", "<", and "+" are verbs in number sentences.

F 13. If the solution set of $x \div 1 = 1 \div x$ is represented on the number line, the representation is the entire number line.

T 14. Parentheses tell us to do the operation within them first.

F 15. $\frac{x}{x} = 1$ is true for all values of x. (Note to teacher: $\frac{0}{0}$ is not possible.)

T 16. $a(3 + 4) = (4 + 3)a$ for all values of a.

F 17. Any mathematical expression consisting of two number phrases is a number sentence.

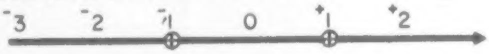
T 18. $(7 + 4)z = z(5 + 6)$ for all values of z.

T 19. A mathematical inequality may also be called a number sentence.

F 20. If $x = {}^{-}10$, then $2x + 8 = 12$.

F 21. Any number sentence may be called an equation.

F 22. The representation shown on the following number line



indicates the set of solutions $\{{}^{-}1, {}^{+}1\}$.

For True-False Items 23-27: Determine if the open sentences are true for the indicated values for the unknown number.

F 23. $x + 0 = 4x$, if $x = 3$

T 24. $\frac{y + 8}{3} = \frac{\frac{16}{y}}{1}$, if $y = 4$

T 25. $5(z + 1) > 5z$, if $z = 3$

T 26. $\frac{z}{5} < \frac{z}{4}$, if $z = 8$

F 27. $(x + y)(x - y) = (x + y)(x + y)$, if $x = 5$ and $y = 2$

Multiple Choice

For Multiple Choice Items 1-3 select the answer which correctly represents the series of operations.

C 1. Multiply the difference of eight and two by three, then add four to the product and divide this sum by two.

A. $8 - 2 \cdot 3 + 4 \div 2$

B. $(8 - 2)3 + 4 \div 2$

C. $\frac{3(8 - 2) + 4}{2}$

D. $\frac{3(8) - 2 + 4}{2}$

E. $3(8 - 2) + (4 \div 2)$

B 2. Double the sum of three and eight and then subtract seven from the product.

A. $2(3) + 8 - 7$

B. $2(3 + 8) - 7$

C. $2(3 + 8 - 7)$

D. $2(3) + (8 - 7)$

E. $2 \cdot 3 + 8 - 7$

A 3. The set of solutions for the open sentence $8 + c < 12$ is:

A. All numbers less than 4.

B. All numbers greater than 4.

C. 4.

D. All numbers except 4.

E. None of the above is correct.

D 4. Which one of the following number sentences is an inequality?

A. $17 > 2 + 2$ is the only inequality.

B. $3 < 2 + 2$ is the only inequality.

C. $x + .10 > 0.1$ is the only inequality.

D. All of the sentences in A, B, and C are inequalities.

E. None of the above answers is correct.

C 5. The phrase $2 \cdot (5) + 4$ represents which one of the following?

A. 8

B. 10

C. 14

D. 18

E. 40

In Questions 6-19 a statement of a mathematical situation has been given in words. Following each are five mathematical phrases or sentences. You are to select the phrase or sentence which best represents the stated situation.

A 6. My age seven years from now. (x = number of years in my present age.)

A. $7 + x$

B. $7 - x$

C. $7(x)$

D. $x - 7$

E. $\frac{7x}{2}$

C 7. The perimeter of a rectangle whose width is one-half its length. (y = number of inches in the width.)

A. $(2y)y$

B. $2y + y$

C. $(2y + y)2$

D. $\frac{1}{2}y + y$

E. $3y$

B 8. The area of a square whose side is s.

A. $2s$

B. $s \cdot s$

C. $4s$

D. $s \cdot 2s$

E. $s + 4$

C 9. A number, plus four times the number, is sixty. (x = the number.)

A. $4x = 60$

B. $x + 4 = 60$

C. $x + 4x = 60$

D. $60 - 4x = 4$

E. $x = 60(4)$

D 10. If I spend one-half the amount of money I have, I will have less than seven dollars. (a = the amount of money I have.)

A. $2a > 7$

B. $\frac{1}{2}a = 7$

C. $a - 2 < 7$

D. $a - \frac{a}{2} < 7$

E. $a > 14$

E 11. I am twice as old as you are. In three years the sum of our ages will be 27. (x = the number of years in your age.)

A. $2x + x = 27$

B. $2x + x = 24$

C. $(2x - 3) + (x + 3) = 27$

D. $(x - 3) + (2x - 3) = 27$

E. $(x + 3) + (2x + 3) = 27$

A 12. In my pocket I have \$3.90. I have a certain number of nickels. I have three times as many dimes as I do nickels and six more quarters than nickels. (n equals the number of nickels.)

A. $5n + 10(3n) + 25(n + 6) = 390$

B. $n + 3n + n + 6 = 390$

C. $5n + 6 = 390$

D. $5n + 10n + 6 = 390$

E. $5n + 10(4 + n) + 25(6n) = 390$

C 13. Ten yards of cloth will cost more than 12 dollars. (x = the cost per yard in dollars.)

A. $x > 12$

B. $x + 10 > 12$

C. $10x > 12$

D. $\frac{10}{x} > 12$

E. $\frac{x}{10} > 12$

E 14. The length of a rectangle is twice its width and its area is 18 square inches. (x = the number of inches in the length.)

A. $2x(x) = 18$

B. $2x + x = 18$

C. $x(x + 2) = 18$

D. $2x = 18$

E. $x(\frac{x}{2}) = 18$

B 15. John is three years less than twice Bill's age. Together their ages are greater than 25 years. (x = number of years in Bill's age.)

A. $(2x + 3) - x > 25$

B. $x + (2x - 3) > 25$

C. $x + (2x + 3) > 25$

D. $x(2x + 3) > 25$

E. $x - (2x + 3) > 25$

C 16. A boat goes downstream 10 miles per hour faster than the rate of the current. Its speed downstream is 25 miles per hour. How fast is the current? (r = rate of current.)

A. $\frac{r}{10} = 30$ D. $r - 10 = 30$

B. $\frac{10}{r} = 25$ E. $r(10) = 25$

C. $r + 10 = 25$

A 17. A two-digit number is 7 more than 3 times the sum of the digits. (t is the ten's digit and u is the unit's digit.)

A. $10t + u = 3(u + t) + 7$

B. $tu - 7 = 3(t + u)$

C. $tu + 7 = 3tu$

D. $10t(u) = 3(t + u)$

E. $10t + u = 3(t + u) - 7$

B 18. A man is 9 times as old as his son. In 3 years the father will be only 5 times as old. What is the age of each? (x = number of years in the son's age.)

A. $9x + 3 = 5x$ D. $12x = 8x$

B. $9x + 3 = 5(x + 3)$ E. $9x = 5x + 3$

C. $9x - 3 = 5(x - 3)$

A 19. The sum of two numbers is 40 and the larger exceeds three times the smaller by 4. (The smaller number is x.)

A. $x + 3x + 4 = 40$ D. $x + 3x = 44$

B. $x + 3x - 4 = 40$ E. $x - 3x + 4 = 40$

C. $x - 4 + 3x = 40$

In Questions 20-23 an open sentence is written. Following are four English sentences. Select the answer which indicates the English sentence or sentences which correspond to the open sentence.

C 20. $y > 10$

(a) I am more than ten years old.

(b) John has ten dollars.

(c) The board is over ten inches long.

(d) There are a number of marbles in the bag.

A. Only (a) and (b) are correct.

B. Only (b) and (c) are correct.

C. Only (a) and (c) are correct.

D. Only (c) and (d) are correct.

E. Only (a), (c), and (d) are correct.

E 21. $x < 360$

(a) A year contains more than 360 days.

(b) 360 degrees is one rotation.

(c) Bob has $3.60.

(d) The area of the square is 360 square inches.

A. Only (a) is correct.

B. Only (b) is correct.

C. Only (c) is correct.

D. Only (d) is correct.

E. None of the answers is correct.

A 22. $K < 13$

(a) The temperature is below thirteen degrees.

(b) Joan bought more than thirteen pairs of shoes.

(c) There are fewer than 13 books overdue.

(d) Bill has thirteen dollars.

A. Only (a) and (c) are correct.

B. Only (b) and (d) are correct.

C. Only (a), (b), and (c) are correct.

D. Only (b), (c), and (d) are correct.

E. All of the answers are correct.

D 23. $3a = 4b$

(a) My salary quadrupled is matched by Jack's salary tripled.

(b) There are four more bananas when the apples are increased by three.

(c) The perimeters of an equilateral triangle and a square are equal.

(d) Three times a certain whole number is equal to four times a number.

A. All of the answers are correct.

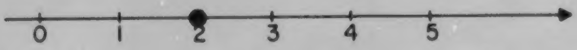
B. Only (a), (b), and (c) are correct.

C. Only (b), (c), and (d) are correct.

D. Only (a), (c), and (d) are correct.

E. None of the answers is correct.

D 24. Which one of the number sentences below has its solution set represented on the number line in the drawing?



(a) $x = 2$ (c) $\frac{x}{2} = 1$

(b) $2x = x + 2$ (d) $x - 2 = 0$

A. Only (a) is correct.

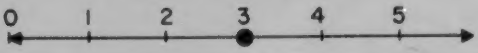
B. Only (b) is correct.

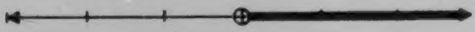
C. Only (c) is correct.

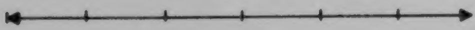
D. All of the sentences are correct.

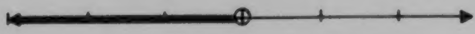
E. None of the sentences is correct.

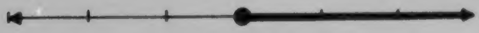
B 25. Which one of the representations on the number line below represents the set of solutions of the number sentence $x > 3$?

A. 

B. 

C. 

D. 

E. 

B 26. Which of the following sets is the solution set of the equation "$x + 7 = 3$"?

A. $\{4\}$ D. $\{10\}$

B. $\{^-4\}$ E. None of the above.

C. $\{4, ^-4\}$

E 27. Which of the following sets is the solution set of the sentence "$1 - x > 2$"?

A. The set of all numbers greater than 3.

B. The set of all numbers less than 1.

C. The set of all numbers greater than $^{-}1$.

D. The set of all numbers greater than $^{-}3$.

E. The set of all numbers less than $^{-}1$.

D 28. Which of the following sets is the solution set of the sentence "$x - 1 < 0$ and $2x - 1 = 3$"?

A. The set of all numbers less than 1.

B. $\{2\}$

C. The set of all numbers between 1 and 2.

D. The empty set.

E. $\{1, 2\}$

C 29. Which of the following sets is the set of solutions for the sentence "$x + 3 > 0$ and $5 - x > 1$"?

A. The set of all numbers between 1 and 3.

B. The set of all numbers greater than $^{-}3$.

C. The set of all numbers between $^{-}3$ and 4.

D. The empty set.

E. The set of all numbers less than 4.

Matching

A. Select the items from the column on the right which match the items in the column on the left. Select an item which best describes the computation shown for each problem. NOTE: The items in the right column may be used once, several times if needed, or not at all. Fill the indicated space on the answer sheet with the letter of your choice.

C 1. A number diminished by 3.

B 2. The temperature rising three degrees.

D 3. The cost of x pencils at 3 cents each.

D 4. A number increased by twice the number.

E 5. The number of yards in x feet.

A. $2x + 1$

B. $x + 3$

C. $x - 3$

D. $x + 2x$

E. $\frac{x}{3}$

B. (Follow directions given for Part A.)

C 6. The number of cents I have if I have t dimes and u pennies.

A 7. A number increased by another number.

E 8. The ratio of two numbers.

A 9. The sum of the digits of a two-digit number.

C 10. Expanded notation for a two-digit number.

A. $t + u$

B. $t - u$

C. $10t + u$

D. tu

E. $\frac{t}{u}$

C. (Follow directions given for Part A.)

A + E 11. John had twenty marbles. He gave some marbles to Bob and he had nine left. How many marbles did he give Bob?

A. $x + 9 = 20$

A + E 12. Mary will be twenty years old nine years from now.

C 13. After losing nine dollars, Allan deposited the rest of his money. The deposit was more than $20. How much money did Allan have?

C 14. From what numbers can you subtract nine so that each difference is greater than twenty?

A + E 15. At eight-thirty, nine more pupils arrived. Then, the total present became 20. How many pupils had arrived before eight-thirty?

B. $x - 20 < 9$

C. $x - 9 > 20$

D. $x + 9 < 20$

E. $20 - x = 9$

Chapter 3

SCIENTIFIC NOTATION, DECIMALS, AND THE METRIC SYSTEM

For this chapter it is assumed that the student has had some acquaintance with the names of numbers, the decimal notation, and finding products involving decimals and per cents.

It is intended that the class exercises be done during class time. The procedure for getting each answer should be discussed before continuing to the next problem.

This chapter should take about 15 days.

3-1. Large Numbers and Scientific Notation

This section seeks to cultivate the ability to read large numbers, an appreciation of them, and ability to write them in scientific notation. Some people prefer to use the term "standard form" instead of "scientific notation." Another name for "scientific notation" is "powers-of-ten" notation, which we have mentioned briefly and used occasionally in the examples. We have tried in the problems to suggest some of the numerous areas in physical science and engineering where this notation finds use.

Your students may be interested to learn that the British have a different way of denoting large numbers. Their words "thousand" and "million" mean the same as ours, but their "billion" means what we would call "a million millions" or "one trillion." They would read the numeral, 3,141,592,653,589,793, as follows:

Three thousand one hundred forty-one billion,
five hundred ninety-two thousand six hundred fifty-three million,
five hundred eighty-nine thousand,
seven hundred ninety-three.

Answers to Class Exercises 3-1a

1. (a) 10^9 (c) 10^{15}

   (b) 10^{12}

2. (a) 7×10^3 (d) 125×10^5

   (b) 5×10^4 (e) 2484×10^6

   (c) 3×10^6 (f) 506×10^6

   It is also correct to have other answers than the above. For example, (b) might be expressed as 50×10^3; 500×10^2, 5000×10^1, or even $50{,}000 \times 10^0$. Some of these answers undoubtedly will appear during the class discussion, and their appearance should be encouraged.

3. (a) 7.6×10 (f) 4.83259×10^3

   (b) 8.59×10^2 (g) 8.412×10^2

   (c) 7.623×10^3 (h) 9.7836×10^3

   (d) 8.463×10^6 (i) 3.412789435×10^6

   (e) 7.648×10 (j) 5.36425×10^{10}

Answers to Class Exercises 3-1b

1. (a) No; because 15 is not between 1 and 10.

   (b) Yes. Fits the definition.

   (c) No; because 0.12 is not between 1 and 10.

2. (a) 5.687×10^3 (c) 3.5×10^6

   (b) 1.4×10

3. (a) 3,700,000 (c) 5,721,000

   (b) 470,000

4. (a) 9.3×10^7
    (b) 1,395,000 or 1.395×10^6
    (c) 91,605,000
    (d) 94,395,000
    (e) 9.1605×10^7 and 9.4395×10^7

Some of the more thoughtful students may wonder why we do not write 93 million, for instance, as 93×10^6 where the exponent is used to indicate the number of zeros in the numeral. There is no point in trying to hide the fact that in many cases this is really a little simpler, and there is no reason to try to prevent students from using it (see comments on Section 2). But two things should be made clear: in the first place, this is the notation which the scientists use; and, second, in the use of logarithms and the slide rule, the scientific notation is certainly much simpler. The students will see some further advantages when they come to the later sections on relative error. The ease of making "order-of-magnitude" estimates when the numbers are expressed in scientific notation is perhaps worth mentioning, also.

Answers to Exercises 3-1

1. (a) 10^3
    (b) 10^{10}
    (c) 10^4
    (d) 10^9
    (e) 10^6
    (f) 10^7
2. (a) 6×10^3
    (b) 6.78×10^2
    (c) 9×10^9
    (d) 4.59×10^8
    (e) 7.8×10^4
    (f) 6×10^3
    (g) 1.56×10^4
    (h) 7.81×10^9
3. 5.06×10^8

4. (a) 100,000
(b) 583
(c) 30,000
(d) 50,000,000
(e) 630
(f) 820,010,000
(g) 436,000,000
(h) 1,732,400

5. (a) Seven hundred eighty-three
(b) Seven million, five hundred thousand
(c) Six hundred thirty-two thousand, seven
(d) Three hundred sixty-two and three hundred sixty-two thousandths
(e) Four billion, two hundred eighty-four thousand, six hundred thirty-two
(f) Four and two thousand five hundred six ten-thousandths

6. (a) $600 = 6 \times 10^2$
(b) $100 = 10^2$
(c) $1200 = 1.2 \times 10^3$
(d) $70900 = 7.09 \times 10^4$
(e) $600,000 = 6 \times 10^5$
(f) $5,362,400 = 5.3624 \times 10^6$

7. 10^2

8. 3.37×10^{17} cu. mi. (If you have mentioned the difference between the British "billion" and the U.S. "billion," your students may be interested to note that a million million is the same in both countries.)

3-2. Calculating with Large Numbers

The objectives of this section are a continuation of those in the first with the added skill of multiplying, using the scientific notation.

Here again, some students may prefer to multiply 93,000,000 by 11,000, for example, by writing the former as 93×10^6 and the latter as 11×10^3, forming the product, 1023×10^9, and then putting it into scientific notation: 1.023×10^{12}. The teacher may prefer this, too, in which case he should use it.

To see that $1.023 \times 10^3 = 1023$, one could refer to the rules for multiplying decimals, or the teacher might prefer to go back to first principles. One way to do the latter would be:

$$1.023 = 1 + \frac{23}{1000} \quad \text{and, since } 10^3 = 1000,$$

$$\begin{aligned} 1.023 \times 10^3 &= (1 + \tfrac{23}{1000}) \times 1000 \\ &= 1000 + \tfrac{23}{1000} \times 1000 \\ &= 1000 + 23 \\ &= 1023. \end{aligned}$$

The student should realize, however, that multiplying a number by 10 is equivalent to moving each digit one place to the left, or simply moving the decimal point one place to the right. He should be able to show on demand why either is so.

Throughout this work with scientific notation it should be emphasized to the students that there are a variety of reasons for using scientific notation. Most of these reasons will not be fully appreciated until much later, when they have had more practice with complex calculations, an introduction to logarithms, and a beginning course in physics. Hence, they may now see no particular advantage for scientific notation in some of the simple examples given here. The main point to be stressed is that, in later work, they will find important uses for it.

Answers to Class Exercises 3-2

1. (a) $60 \times 186{,}000$

   (b) $60 \times 60 \times 186{,}000$

   (c) $24 \times 60 \times 60 \times 186{,}000$

   (d) $365 \times 24 \times 60 \times 60 \times 186{,}000$

   (e) Do not "round" until multiplications have been performed; if "rounding" takes place earlier, the result becomes less accurate. Before rounding, it is 5,865,696,000,000.

   (f) trillion

   (g) The speed of light used is the result of rounding an approximation. The number of days in a year (365) is an approximation also.

Answers to Exercises 3-2

1. (a) 6×10^{10}

   (b) 1.2×10^{19}

   (c) 3.5×10^{13}

   (d) 6×10^{8}

   (e) 7.63×10^{7}

   (f) 2.16×10^{5}

   (g) 2.1×10^{6}

   (h) 9.3×10^{14}

2. (a) 6.3×10^{11}

   (b) 10^{9}

   (c) 4.65×10^{9}

   (d) 1.1×10^{9}

3. (a) 1.728×10^{4} miles

   (b) 7.2×10^{4} miles

   (c) 4.32×10^{6} miles

   (d) 6.3072×10^{7} miles

   No, this distance is about 93 million miles (or 9.3×10^{7} miles).

4. (a) 4×10^7 (c) 1.2×10^6
   (b) 2×10^{10}

5. 7.2×10^9 miles

6. Yes, you could have made 63,072,000 marks.

7. About 6.132×10^8 miles or 613,200,000 miles.

In Problem 3 on the previous page, notice that the speed of the space ship is about a mile per second, which is about 32 million miles a year.

3-3. Calculating with Small Numbers

Answers to Class Exercises 3-3

1. (a) 10^{-4} (d) 10^{-9}
   (b) 10^{-6} (e) 10^{-5}
   (c) 10^{-7}

2. (a) $\frac{1}{10^3}$ or $\frac{1}{1000}$ (c) $\frac{1}{10^7}$ or $\frac{1}{10,000,000}$
   (b) $\frac{1}{10^5}$ or $\frac{1}{100,000}$ (d) $\frac{1}{10^6}$ or $\frac{1}{1,000,000}$

3. (a) 10^{-3} (c) 10^{-9}
   (b) 10^{-6} (d) 10^{-12}

Answers to Exercises 3-3

1. (a) 9.3×10^{-2}
   (b) 10^{-4}
   (c) 10^{-6}
   (d) 10^{0}
   (e) 6.21×10^{-3}
   (f) 10^{-2}
   (g) 7.006×10^{-1}
   (h) 9.07×10^{-7}
   (i) 6×10^{0}
   (j) 4.5×10^{-3}

2. (a) 0.000093
   (b) 0.107
   (c) 0.000001
   (d) 0.0005
   (e) 0.007065
   (f) 0.1
   (g) 0.00000143
   (h) 0.00038576

3. (a) 6.3×10^{5}
   (b) 1.57×10^{-4}
   (c) 2.4×10^{-6}
   (d) 5.265×10^{-5}
   (e) 3.6235×10^{2}
   (f) 4.32×10^{-3}
   (g) 3.05×10^{-9}
   (h) 6.95×10^{0}

4. (a) $^{-}3$
   (b) 6.3
   (c) $^{-}7$
   (d) 5
   (e) $^{-}1$
   (f) $^{-}3$
   (g) 21300
   (h) 0.0000213

5. The number zero, since a product is zero only when one of the factors is zero. Neither a "power of ten" nor "a number between 1 and 10" can be zero.

3-4. Multiplication: Large and Small Numbers

Answers to Exercises 3-4

1. (a) 10^{-7}
   (b) 3×10^{-3}
   (c) 10^{-13}
   (d) 8×10^{-5}
   (e) 7×10^{-7}
   (f) 5.7×10^{-10}
   (g) 10^{24}
   (h) 10^{-3}

2. (a) 2.88×10^{-8}
   (b) 5.4×10^{-9}
   (c) 1.4×10^{-7}
   (d) 3×10^{-10}
   (e) 4.56×10^{-6}
   (f) 5.6896×10^{-6}

3. (a) 10^{2}
   (b) 10^{2}
   (c) 10^{-6}
   (d) 10^{-7}

4. 3.4594×10^{-1}

5. 4.125×10^{5} dollars

6. 9.939×10^{9} dollars

7. 1.08×10^{6} grams

8. 1.66×10^{-15} grams

3-5. Division: Large and Small Numbers

Answers to Class Exercises 3-5

1. (a) 10^{-9} (b) 10^5

2. (a) 10^5 (c) The same number
   (b) Because 7 - 2 equals 5

3. (a) 9 (c) Both equal 10^9
   (b) 10^9

4. (a) 10^{-9} (b) Yes. Both equal 10^{-9}.

5. Yes. Each side equals 10^5.

6. $10^{m - n}$

7. (a) 10^{16} (c) 10^{-12}
   (b) 10^1

8. Yes. Yes.

9. (a) 3×10^{-11} (c) 3×10^7
   (b) 2×10^{-1}

Answers to Exercises 3-5

1. (a) 10^3 (e) 10^{-2}
   (b) 10^2 (f) 10^{-10}
   (c) 10^{10} (g) 10^{-6}
   (d) 10^5 (h) 10^{-1}

2. (a) 10^{7}
(b) 10^{4}
(c) 10^{18}
(d) 10^{29}
(e) 10^{24}
(f) 10^{30}
(g) 10^{18}
(h) 10^{7}

3. (a) 10^{-7}
(b) 10^{-4}
(c) 10^{-18}
(d) 10^{-29}
(e) 10^{-24}
(f) 10^{-30}
(g) 10^{-18}
(h) 10^{-7}

4. (a) 10^{-3}
(b) 10^{-10}
(c) 10^{2}
(d) 10^{6}
(e) 10^{-2}
(f) 10^{-5}
(g) 10^{10}
(h) 10^{1}

5. (a) 2×10^{-3}
(b) 7×10^{-7}
(c) 1.2×10^{9}
(d) 2.4×10^{2}
(e) 4×10^{-2}
(f) 4×10^{-3}

6. (a) $^{-}2$ and $^{-}1$
(b) $^{-}2$ and $^{-}1$
(c) 2, $^{-}2$ and $^{-}3$
(d) $^{-}2$ and 0
(e) $^{-}2$, 2 and 4
(f) $^{-}2$, 3, $^{-}2$ and 5

7. 9.2×10^{7}. Treat as problem in equations,

$$\frac{3}{100}y = 2.76 \times 10^{6}.$$

8. No, since it will take 100 years to spend this sum of money.

9. About 1180 days (rounded to nearest ten).

10. \$20,000. Treat as problem in ratio.

11. 40%. Treat as problem in equations,

$$14 \times 10^6 = \frac{y}{100}(35 \times 10^6).$$

12. (a) The mass of the proton is greater.

    (b) 1.8×10^3

3-6. Use of Exponents in Multiplying and Dividing Decimals

Note that here division is handled somewhat differently from the discussion in Chapter 9 of Volume I. Powers of 10 are used in such a fashion that in the formal division portion a whole number is divided by a whole number. Many people find this easier to follow than the procedure which uses a decimal as the dividend.

There are too many problems in the exercises to be given as one assignment. Some of the students may need the extra practice and will need two or more days.

Answers to Class Exercises 3-6

1. (a)

$$\begin{array}{rcrcl} 6.14 & = & 614 & \times & 10^{-2} \\ \times\ \underline{0.42} & = & \times\ \underline{42} & \times & \underline{10^{-2}} \\ & & 1228 & & \\ & & \underline{2456\ } & & \\ & & 25788 & \times & 10^{-4} = 2.5788 \end{array}$$

(b)

$$\begin{array}{rcrcl} 0.625 & = & 625 & \times & 10^{-3} \\ \times\ \underline{0.038} & = & \times\ \underline{38} & \times & \underline{10^{-3}} \\ & & 5000 & & \\ & & \underline{1875\ \ } & & \\ & & 23750 & \times & 10^{-6} = 0.023750 \end{array}$$

(c)

$$\begin{array}{rcrcl} 649.3 & = & 6493 & \times & 10^{-1} \\ \times\ \underline{14.68} & = & \times\underline{1468} & \times & \underline{10^{-2}} \\ & & 51944 & & \\ & & 38958\ \ & & \\ & & 25972\ \ \ \ & & \\ & & 6493\ \ \ \ \ \ \ & & \\ \hline & & 9531724 & \times & 10^{-3} = 9531.724 \end{array}$$

(d)

$$\begin{array}{rcrcl} 11.4 & = & 114 & \times & 10^{-1} \\ \times\ \underline{0.0031} & = & \times\ \underline{31} & \times & \underline{10^{-4}} \\ & & 114 & & \\ & & \underline{342\ \ } & & \\ & & 3534 & \times & 10^{-5} = 0.03534 \end{array}$$

Answers to Exercises 3-6

1. (a) 0.18063
   (b) 0.0684
   (c) 7500
   (d) 399.529
   (e) 7.2

2. (a) $^-2$
   (b) $^-3$
   (c) 4
   (d) $^-3$
   (e) 63700
   (f) 2, 1
   (g) 0.0412

3. (a) 300200

(b) 6.1

(c) 175.02

(d) 0.000007

(e) 160

4. (a) $135 \times 6 \times 10^{-2} = 810 \times 10^{-2} = 8.1$

(b) $(76 \times 10^{3}) \times (3 \times 10^{3}) = 228 \times 10^{6} = 228{,}000{,}000$

(c) $(18 \times 10^{3}) \times (3 \times 10^{-4}) = 54 \times 10^{-1} = 5.4$

(d) $(35 \times 10^{-4}) \times (16{,}301 \times 10^{-3}) = 570535 \times 10^{-7} = 0.0570535$

(e) $(6 \times 10^{6}) \times (275 \times 10^{-4}) = 1650 \times 10^{2} = 165{,}000$

(f) $(7 \times 10^{-2}) \times (3 \times 10^{2}) \times (2 \times 10^{-2}) \times (6 \times 10^{3})$

$= 252 \times 10^{1} = 2520$

5. (a) $(63 \times 10^{-1}) \div (3 \times 10^{-1}) = 21$

(b) $(78 \times 10^{-2}) \div (13 \times 10^{0}) = 6 \times 10^{-2} = 0.06$

(c) $\dfrac{8750 \times 10^{0}}{875 \times 10^{-2}} = 10 \times 10^{2} = 1000$

(d) $\dfrac{1470 \times 10^{-4}}{75 \times 10^{-2}} = 19.6 \times 10^{-2} = 0.196$

(e) $27 \times 10^{-2}\,\overline{)\,84402 \times 10^{-5}} = 3126 \times 10^{-3} = 3.126$

(f) $18 \times 10^{2}\,\overline{)\,216 \times 10^{-1}} = 12 \times 10^{-3} = 0.012$

6. $\dfrac{4186 \times 10^{-1} \times 19 \times 10^{-3}}{13 \times 10^{-2}} = 6118 \times 10^{-2} = 61.18$

7. $\frac{6 \times 840}{.04} = \frac{5040 \times 10^0}{4 \times 10^{-2}} = 1260 \times 10^2 = 126000$

It is necessary to make 126,000 pieces.

8. $\frac{5\frac{1}{3} \times (5.9 \times 10^{12})}{100{,}000 \times (3.2 \times 10^7)} = \frac{\frac{16}{3} \times (59 \times 10^{11})}{10^5 \times 32 \times 10^6} = \frac{16 \times 59}{32 \times 3} = 9.8$

It will take about 9.8 years to go one way, 19.6 years round trip.

3-7. The Metric System: Metric Units of Length

"The invention of the Hindu-Arabic decimal number system is one of man's outstanding achievements. With it, for the first time in history, masses were able to learn the art of computation. Later Simon Stevin still further simplified the processes of computation by the introduction of the decimal fraction. Today, the decimal fraction should be called the common fraction, so widely is it used in commerce and technology.

"Still later came the metric system of measures, based upon the units, meter, liter, and gram, which are also decimal. ...If the selection of a system of measures were optional with educators, they would unhesitatingly choose a decimal system. They are well aware of the tremendous efforts required to learn, for example, the relationship between linear units in our system: 1 inch $= \frac{1}{12}$ foot, 1 foot $= \frac{1}{3}$ yard, 1 yard $= \frac{2}{11}$ rod, 1 rod $= \frac{1}{320}$ mile. In contrast, they appreciate the simplicity and ease with which the pupil could learn: 1 millimeter = 0.1 centimeter, 1 centimeter = 0.01 meter and 1 meter = 0.001 kilometer.

[pages 137-142]

"From the point of view of teaching and learning, it would not be easy to design a more difficult system than the English system. In contrast, it would seem almost impossible to design a system more easily learned than the metric system."[1]

The above quotation from the Twentieth Yearbook of the National Council of Teachers of Mathematics served as a motivation for this section.

The brief historical sketch is intended to help the pupils see the decimal foundation and origin of the metric system.

Although the pupils have had a brief introduction to the metric system in the 7th grade, we urge you to have them "live metric" for a few days. Encourage them to think in the metric system. The pupils should have a metric ruler at hand in order to get well acquainted with the linear units. We suggest also that you use the metric system frequently in subsequent chapters in which measurements are required.

Because of the length of this chapter we were not able to introduce the centigrade scale for temperature measurements. You will also note that we have not developed in any detail the two systems now in vogue, namely, the MKS (meter, kilogram, second) or the CGS (centimeter, gram, second). As a matter of fact, we borrowed from both systems in order to make our tables more easily understandable.

We highly recommend the Twentieth Yearbook of the National Council of Teachers of Mathematics as a rich reference text for both your professional library and your pupils' mathematics library. The article in Popular Mechanics, December 1960, titled "The Metric System--Pro and Con" was highly recommended by many teachers.

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[1] Clark, John R., "A Note on the Yearbook," The Metric System of Weights and Measures, Twentieth Yearbook of National Council of Teachers of Mathematics, Bureau of Publications, Teachers College, Columbia University, New York, 1948.

Answers to Exercises 3-7a

1. (a) 10
   (b) 100
   (c) 1,000
   (d) 10,000
   (e) 100,000
   (f) 1,000,000

2. (a) 1.111111
   (b) 534.2
   (c) .24536
   (d) 564
   (e) 6.043278
   (f) 202020.2
   (g) 15

3. (a) 500
   (b) 2000
   (c) 0.5
   (d) 25.4
   (e) 1500
   (f) 325
   (g) 3.5
   (h) 4.74
   (i) 55
   (j) 625

4. (a) 4×10^{7}
   (b) 4×10^{4}
   (c) 4×10^{10}

5. (a) 1.3×10^{-5} m.
   (b) 2.34×10^{-2} m.
   (c) 6.730×10^{6} m.
   (d) 6.94×10^{-4} m.
   (e) 1 m.

In connection with the introduction of the micron, your students may be interested in working with a still smaller unit the angstrom unit which is defined as 10^{-10} m, or

$$1 \text{ angstrom unit} = 10^{-10} \text{ m.}$$

An atom has a linear dimension of about 2×10^{-10} m. = 2 angstroms, and this suggests why angstroms are so useful in studies of atomic structure and in sub-atomic physics.

Answers to Exercises 3-7b

1. (a) 100 m. ≈ 110 yds.
   (b) 200 m. ≈ 220 yds.
   (c) 400 m. ≈ 440 yds.
   (d) 800 m. ≈ 880 yds.
   (e) 1500 m. ≈ 1650 yds.

Note: The careful student may remark that the approximate answers in (d) and (e), obtained as multiples of the 100 m. equivalent, are not correct to the nearest 10 yds. For, if we use a more careful estimate, we see that

$$1 \text{ m.} = \frac{39.37}{36} = 1.0936 \text{ yds.}$$

and, hence,

$$100 \text{ m.} \approx 109.36 \text{ yds.}$$

Using this as a basis for the longer distances we see that 800 m. ≈ 870 yds. and 1500 m. ≈ 1630 yds. However, for the purposes of this section, either method of approximating should be acceptable.

   (f) 1500 m. ≈ 0.93 miles.
   (g) 10 km. ≈ 6.2 miles
   (h) 100 km. ≈ 620 miles

2. (a) From $1 \text{ m.} = 39.37 \text{ in.} = \frac{39.37}{12} \text{ ft.}$

we see that $1 \text{ ft.} = \frac{12}{39.37} \text{ m.}$

Hence, $29{,}000 \text{ ft.} = \frac{12 \times 29000}{39.37} = \frac{12 \times 29 \times 10^3}{3937 \times 10^{-2}}$

or

$$29{,}000 \text{ ft.} \approx \frac{12 \times 29}{3937} \times 10^5 \approx 8.8 \times 10^3 \text{ m.}$$

(b)

$$34{,}200 \text{ ft.} \approx \frac{12}{39.37} \times 34{,}200 \text{ m.}$$
$$\approx 1.042 \times 10^4 \text{ m.}$$

3. (a) From $1 \text{ km.} \approx 0.62$ miles, we have

$$1 \text{ mile} \approx \frac{1}{0.62} \text{ km.} \approx 1.61 \text{ km.}$$

It is probably better practice, however, to go back to the fundamental equivalent 1 m. = 39.37 in. and work it through from this beginning, as is done in the text for the kilometer in terms of miles.

(b)

$$9.29 \times 10^7 \text{ miles} \approx 14.96 \times 10^7 \text{ km.}$$
$$\approx 1.5 \times 10^8 \text{ km.}$$

4. 21.6 cm. × 27.9 cm.

6. 100 ft. per sec. = 100 × 12 × 2.54 cm. per sec. = 3048 cm. per sec. Hence, 100 ft. per sec. is the faster.

3-8. Metric Units of Area

Answers to Exercises 3-8

1. (a) 10^{-4} sq. m.
   (b) 10^{-6} sq. m.
   (c) 10^2 or 100 sq. mm.

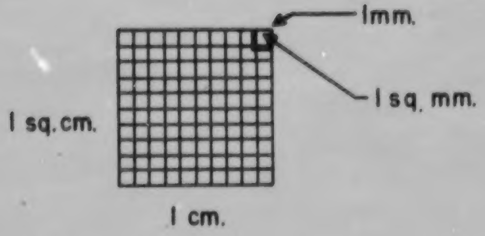
Answers to Exercises 3-8 (Continued)

(d) 10^4 sq. cm.

(e) 10^{-6} sq. km.

Note: If the pupils prefer to write these numbers in expanded form, this is satisfactory.

2.



Note: This sketch is not drawn to scale. The pupils should be encouraged to draw such a sketch also, simply to emphasize the number of sq. mm. in a sq. cm. A comparison of this sketch with 1 sq. cm. drawn to scale might then be useful.

3. (a) 322 sq. cm.
   (b) 12.768 or 13 sq. m.
   (c) 3589 sq. cm. or 0.3589 sq. m.
   (d) 15 sq. mm. or 0.15 sq. cm.

4. 12.768 sq. m. = 127,680 sq. cm.

5. 113.04 sq. m.

6. 2.56 sq. m.

7. 29 cm.

8. (a) 8354 sq. cm. (b) 6.45 sq. cm.

10. (a) 57.5×10^6 sq. miles or 5.75×10^7 sq. mi.
    (b) 1.39×10^8 sq. mi.

3-9. Metric Units of Volume

Answers to Exercises 3-9

1. (a) 1000 (c) $\frac{1}{1,000,000,000}$

   (b) $\frac{1}{1,000,000}$ (d) 10^{18}

2. 352.8 cu. cm.

3. 51.1 cu. cm. or 51,100 cu. mm.

3-10. Metric Units of Mass and Capacity

You will notice that without additional comment we have referred to the gram as the unit of mass, not weight, in the metric system. It was thought best not to involve the pupils in a full-scale discussion of this point here. An adequate treatment of the ideas belongs in a high school physics course. It is likely, however, that some of your pupils will ask questions about this terminology, and you will want to know how to answer them correctly, if not in complete detail. The following discussion should be adequate for this purpose. Should you wish more information on the subject, you might refer to Physics, Volume I, prepared by the Physical Science Study Committee of Educational Services, Inc. (This is the first volume of the so-called M.I.T. course for high school physics students.)

What is the weight of an object? It is a measurement of the force or "pull," of gravity on that object. An ordinary bathroom scale measures this pull by the amount it stretches, or twists, a spring. We think of weight as measuring the "quantity of matter" in an object, in some sense. A box of lead weighs more than the same box filled with feathers because the lead has a greater "quantity of matter" packed into the given volume than do feathers. There is another way to measure the "quantity of matter" of an object. This is to compare the object with some standard, or unit, body on a balance. If we have a supply of identical objects called "grams" we can determine the number of these "grams" it takes to balance the box of lead. This number of grams we call the mass of this much lead.

These two different ways of measuring "quantity of matter" can be used interchangeably for most purposes in any one fixed location, but they are not, strictly speaking, measurements of the same thing. Weight depends on the nearness to the center of the earth. The pull of the earth--gravity--on the box of lead would be much smaller in a space ship as far from the earth as is, say, the moon. The weight of the lead would be much smaller there. However, the lead would balance the same number of "grams" on the space ship which it balanced on the earth (the "grams" would themselves weigh correspondingly less) so its mass would be unchanged. To summarize:

> Weight is the pull of the earth. It changes as the distance between the object and the center of the earth changes.
>
> Mass is a comparison of the object with a set of unit bodies. It does not depend on the position in space where it is measured.

We humans are normally restricted to a very narrow range of altitude above sea level, and, with that restriction, we can think of weight and mass as having a definite fixed relationship. (It is tempting to predict that as we enter the space age and are released from these restrictions, weight and mass and the distinctions between them will become subjects for household discussion.) In the English system weight is measured in pounds, mass in slugs. An object which has a mass of 1 slug has a weight of approximately 32.2 pounds at sea level. The weight in pounds of any object is approximately 32.2 times its mass in slugs. The more common unit in this system, of course, is the pound. In the metric system, when mass is measured in grams, weight is measured in dynes. An object whose mass is one gram has a weight of approximately 980 dynes at sea level. The weight of this same object in the English system would be approximately 0.0022 pounds. As you probably know, the more familiar unit in the metric system is the unit of mass, the gram.

In the text we have introduced only the more familiar units, pound and gram. This has made it necessary to use both words, mass and weight. You must judge for yourself how much of the above discussion of the two ideas you will use in your classroom. If the subject does come up, however, be sure to make one point: both mass and weight can be measured in either of the systems of units, English and metric. If you fail to point this out to the pupil, he may interpret the discussion in the text to mean that weight is something measured in the English system and mass something measured in the metric system.

The distinction between mass and weight has to be kept clearly in mind in speaking of the definition of pound in terms of the metric standard kilogram. We say a pound is defined to correspond to 0.45359237 kilograms--that is, the pound (weight) is the weight of a mass of 0.45359237 kilograms. Actually, it is quite correct to speak of a lb. (mass) and a lb. (weight) so long as the distinction is clearly made. Thus, one may, if he wishes, write 1 lb. (mass) = 0.45359237 kg. We shall not do this here. In the unabridged dictionary, however, a pound is defined as a unit of mass or of weight.

Your classes may be interested in the agreement which went into effect on July 1, 1959, and created (for the first time) an international yard and an international pound. The six English-speaking nations (U.S.A., United Kingdom, Canada, Australia, Union of South Africa, New Zealand) agreed to standardize, as of this date, their definitions of yard and pound. The definition 1 in. = 2.54 cm. dates from this agreement and so also does the above definition for the pound. Thus, we now have an international yard = 0.9144 meters and an international pound corresponding to 0.45359237 kilograms (i.e., the pound weight corresponds to the weight of a mass of 0.45359237 kilograms). Prior to this agreement the U.S. inch ≈ 2,540005 cm. and the British inch ≈ 2.539996 cm., as a result of shrinkage in the British prototype bar. The situation with the pound was even more confused, since, prior to 1959,

1 U.S. lb. ≈ 453.5924277 gms.
1 British lb. ≈ 453.59233 gms.
1 Canadian lb. ≈ 453.59237 gms.!!

No agreement could be reached for an international gallon. Hence, we still have the U.S. gallon defined as 231 cu. in. and the British Imperial gallon = 1.20094 U.S. gallons.

Although the metric and the English systems of measure are the major systems, your pupils may be interested in looking in the unabridged dictionary under "measure" to see the great number of other measures used in countries throughout the world.

Answers to Exercises 3-10

1. (a) 352.8 grams (b) .3528 kilograms

2. (a) 673.5 milliliters (b) .6735 liters

3. (No allowance is made for the approximate nature of measures.)

   (a) $81 \times 81 \times 81 = 531{,}441$ (cu. in.)

   (b) $\frac{531{,}441}{1728} = 307.5$ (cu. ft.)

   (c) $307.5 \times 62.4 = 19{,}188$ (lb.)

4. (a) $(2.06)^3 = 8.741816$ (cu. m.)

   (b) 8,741.816 liters

   (c) 8,741.816 kilograms

5. Problem 4 should take less time because only one set of computations needs to be made.

   Also, one more measure (liters) is found.

   To convert a number of cu. in. to gallons, a further computation would have to be made. There are 231 cu. in. in a gallon.

$$\frac{531,441}{231} \approx 2300.6 \text{ (gallons)}.$$

6. (a) 2500 milliliters

   (b) 2500 grams = 2.5 kgm.

   (c) .0025 metric tons

7. (a) 27000 cu. cm. or 2.7×10^4 cu. cm.

   (b) 2.7×10^1 liters or 27 liters

   (c) 27 kilograms

8. (a) $3.37 \times 10^{17} \times (1.6)^3$ cu. km.

   (b) $(3.37) \times 10^{17} \times (1.6)^3 \times (10^5)^3$ cu. cm.

   or $(3.37) \times (1.6)^3 \times 10^{32}$ cu. cm.

   The eager pupil may carry this to the form

$$13.8 \times 10^{32} \approx 1.4 \times 10^{33} \text{ cu. cm.}$$

9. (a) 6 U.S. gallons

   (b) 60 Imperial gallons

Research Problem:

(c) Gold is weighed in troy units, while feathers are weighed in avoirdupois units. The pound of feathers weighs more.

Sample Questions

The following sample questions are not intended as a single test on Chapter 3 material. They are included simply to aid the teachers in making up test questions.

True-False

F. 1. An exponent tells us by what number the base is multiplied.

F 2. $10^2 \cdot 10 = 10^2$

T 3. $10^5 \cdot 10^{-5} = 1$

T 4. $1{,}000{,}000 = 10^6$

T 5. $1\frac{1}{2}\% = 1.5 \times 10^{-2}$

T 6. $\frac{1}{10^5} = \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$

T 7. 2.76 billion $= 2.76 \times 10^9$

F 8. $10^3 = 10^{-2} + 10^5$

F 9. $93{,}000{,}000 = 9.3 \times 10^6$

F 10. $10^3 \cdot 10^3 = 100^6$

F 11. 11×10^7 is written in scientific notation.

Other Sample Questions

1. Write the following using scientific notation.

4×10^{-2} (a) 0.04

5.68×10^3 or 5.680×10^3 (b) 5680

$\underline{6.13207 \times 10}$ (c) 61.3207

$\underline{2.8 \times 10^{7}}$ (d) 28 million

$\underline{4.95 \times 10^{3}}$ (e) 0.00495×10^{6}

$\underline{2 \times 10^{-4}}$ (f) 0.2×10^{-3}

2. Perform the indicated operations. Write answers in scientific notation.

$\underline{5 \times 10^{6}}$ (a) $(5 \times 10^{-6}) \times (10^{12})$

$\underline{1.8 \times 10^{-7}}$ (b) $(6 \times 10^{-5}) \times (3 \times 10^{-3})$

$\underline{7 \times 10^{2}}$ (c) $19.6 \div 0.028$

$\underline{3.935 \times 10^{-6}}$ (d) $0.787 \div 200{,}000$

3. Fill in the blanks with the proper symbol.

$\underline{\quad -2 \quad}$ (a) $7.81 = 781 \times 10^{\square}$

$\underline{\quad 2 \quad}$ (b) $61320 = 613.20 \times 10^{\square}$

$\underline{\quad -2 \quad}$ (c) $1\% = 1 \times 10^{\square}$

$\underline{\quad 3 \quad}$ (d) $7.60 \times 10^{3} = 7.600 \times 10^{\square}$

4. Place the proper symbol, × or ÷, in the ☐ to make the following true.

$\underline{\quad \div \quad}$ (a) 100 mm. = 100 m. ☐ 10^{3}

$\underline{\quad \div \quad}$ (b) 1 cm. = 1 m. ☐ 10^{2}

×		(c) 3 km. = 30 m. ☐ 10^2
×		(d) 63 m. = 6.3 cm. ☐ 10^3
×		(e) 1 m. = 1 mm. ☐ 10^3
$^{-}1$	5.	To find 10 % of a number one may multiply the number by the ___?___ power of 10.
Divide	6.	To change from a certain standard unit (in the metric system) of measure to a larger one it is necessary to ___?___ (multiply or divide?) by a positive power of 10.
Six thousand one hundred forty	7.	Write in words the value of 6143 after it has been rounded to the nearest ten.
10^{-9}	8.	Express $(\frac{1}{1000})^3$ in scientific notation.
Subtract	9.	To divide 10^a by 10^b where a and b are integers, it is necessary only to ___?___ b from a and use this result as an exponent of 10.

Chapter 4

CONSTRUCTIONS, CONGRUENT TRIANGLES, AND THE PYTHAGOREAN PROPERTY

This chapter should require 13 to 15 days.

Topics considered in this chapter are drawings, constructions, symmetry, congruence, and the Pythagorean Property. A primary goal is to provide for instruction about the Pythagorean Property. However, an equally important goal is to extend the students' experience with geometric relations through an introduction to informal deduction. The study of congruence and its applications provides an excellent way of doing this. As an introduction to congruence, constructions with compass and straightedge are considered. Since drawings or even sketches are often very useful in mathematics and its applications, a treatment of this topic is introduced before the restrictions imposed by traditional constructions in the Euclidean sense are to be studied. In Section 4-9 students are asked to draw or sketch several solids, in order to provide additional experiences in drawing figures and also to continue the simultaneous treatment of plane and solid geometry introduced in Volume I of *Mathematics for Junior High School*.

In an earlier edition of Volume II, constructions were treated in a separate chapter. Because of the close relationship of the sections on constructions to congruence, however, it has seemed desirable to treat these topics in closer association with each other. Quite a number of teachers who taught the experimental edition recommended this change. Some teachers were also concerned about teaching the Pythagorean Property before the treatment of irrational numbers, now to be found in Chapter 6. In the preparation of this edition it was decided that it was more appropriate to first introduce numbers like $\sqrt{13}$ and $\sqrt{85}$, as they are needed in applications involving the Theorem of Pythagoras, rather than to introduce them through the more abstract considerations of the real number system.

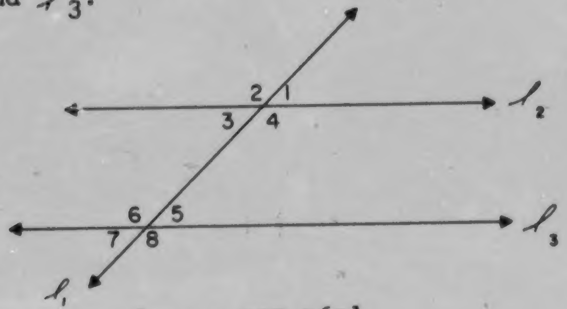
It probably will be desirable to review Chapter 10 in Volume I along with, or as an introduction to, this chapter. Particularly, the sections on parallels and parallelograms include properties which will be needed in some of the problems.

4-1. Introduction to Mathematical Drawings and Constructions

This section gives the pupils an opportunity to use the protractor and ruler as well as some of the other tools which are helpful in making accurate drawings. The students have an opportunity to find out more about the tools used by draftsmen. A secondary dictionary gives a brief description of these tools, and an unabridged dictionary gives a better explanation. Students should be encouraged to find more complete descriptions. French curves are now available in geometry tools that can be purchased in most ten-cent stores and 30-60 and 45-45 triangles are readily available. Some might be brought to class by some students. A pantograph is a relatively simple tool; some of the boys (or girls) might be interested in trying to construct one. Webster's unabridged dictionary gives an explanation that should be adequate for this purpose.

There are two expressions which the pupils have had before that they may have forgotten. They are: transversal and corresponding angles. It may be necessary to review these before the pupils use them.

(1) Transversal. A transversal is a line that cuts two or more lines. In the figure, ℓ_1 is a transversal cutting ℓ_2 and ℓ_3.



[pages 157-160]

(2) Corresponding Angles. When a transversal cuts two lines, as in the figure on the previous page, eight angles are formed. This gives four sets of corresponding angles containing two angles to the set. Angles are said to be corresponding when they lie on the same side of the transversal and on the same relative side of the lines. In the figure we have the following sets of corresponding angles.

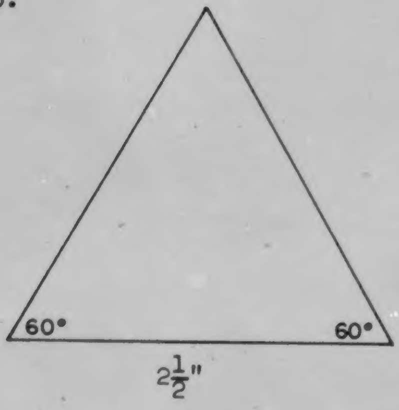
$\angle 4$ and $\angle 8$
$\angle 1$ and $\angle 5$
$\angle 3$ and $\angle 7$
$\angle 2$ and $\angle 6$

Answers to Exercises 4-1

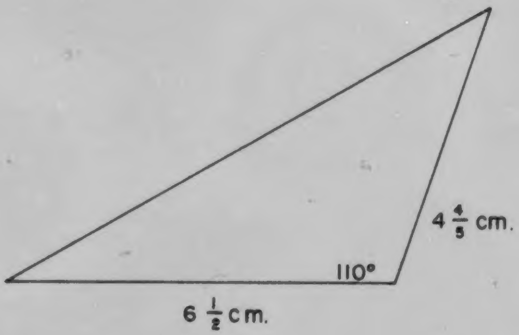
3. a, c, e, and h are acute angles.
d, f, and g are obtuse angles.

4. (a) 4 pairs
(b) 4 pairs
(c) 160 or 200
(d) yes

5.



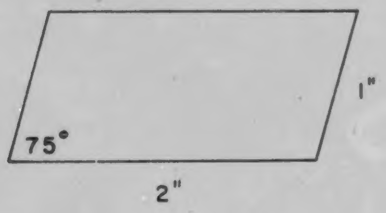
6. obtuse



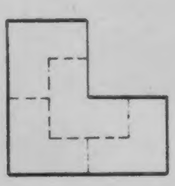
7. Yes; yes.
   (a) 90°, 90°
   (b) yes
   (c) 180, 180, $m(\angle A) + m(\angle B) = 180$, $m(\angle B) + m(\angle C) = 180$, $m(\angle C) + m(\angle D) = 180$, $m(\angle D) + m(\angle A) = 180$.

8. $m(\angle X) + m(\angle Y) = 180$
   $m(\angle Y) + m(\angle Z) = 180$
   $m(\angle Z) + m(\angle R) = 180$
   $m(\angle R) + m(\angle X) = 180$

9.



10.



4-2. Basic Constructions

This section deals entirely with basic classical geometric constructions. Students will need instruction on the correct usage of compasses. The compass should be held so as to keep the radius constant. This is particularly true since most students will be using inexpensive compasses that do not lock in position and there is a tendency for the radius to slip. Most of the inexpensive compasses used by students today have an inch scale; it is worthwhile to point out that this is not accurate since the position of the pencil point can be raised or lowered with a resulting change in radius. If a radius is to be some measure given in terms of a standard unit, it should be checked against a ruler. This is probably the first time that many students have thought of a compass as an instrument for measuring lengths. It should be stressed that a compass will measure equal lengths but will not measure in terms of inches. Of course it is possible to use the ruler to draw a segment with the length of a standard unit of measure and then copy this segment using the compass.

Four constructions are given. The first, measuring equal segments, is usually assumed; but since this is a new concept for students of this age, the construction is given in detail. The constructions for bisectors and copying an angle do not deviate from the traditional; the vocabulary is consistent with modern terminology. The constructions for perpendiculars are given in Section 4-5 at a time when the students can show that the constructions will actually produce perpendiculars.

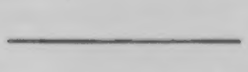
When it is possible to present a construction easily without lettering the drawing, this is done. This permits a more general description of the construction. At times the English statement becomes so cumbersome that lettered drawings are used to simplify the statement.

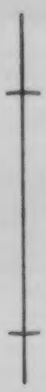
Euclid tried to develop his geometry using as few postulates as possible. The postulates on lines and circles were stated in such a way that it now has become traditional in Euclidean geometry to restrict constructions to the use of the compass and straightedge, so that these constructions may be based on his limited number of postulates.

The French mathematician of the 19th century, Peaucellier, usually is given credit for developing the first instrument with which a straight line can be drawn without the use of a straightedge. His instrument is a type of linkage much like a pantograph. The proof that the instrument does what is claimed for it, is based on the transformation of inversion. See any text on college geometry.

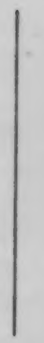
Answers to Exercises 4-2a

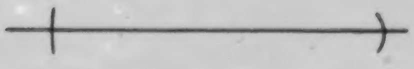
1.



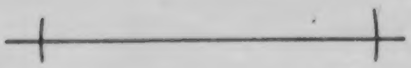


2.

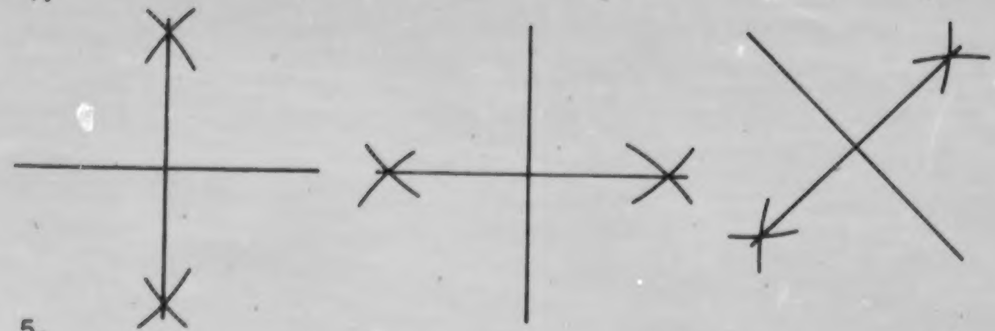




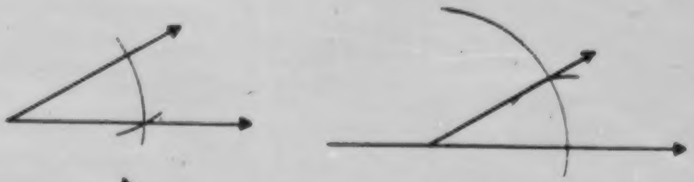
3.

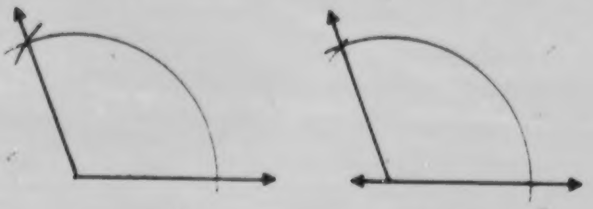


4.

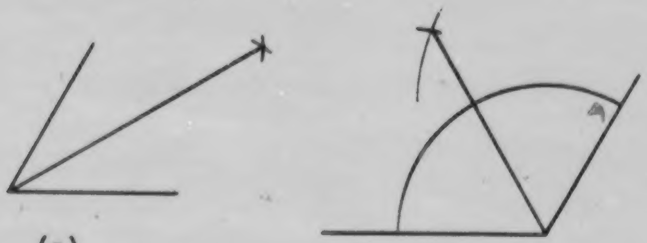


5.

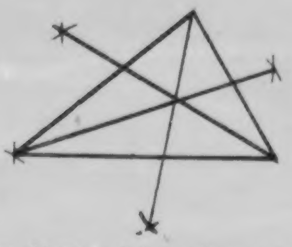




6.

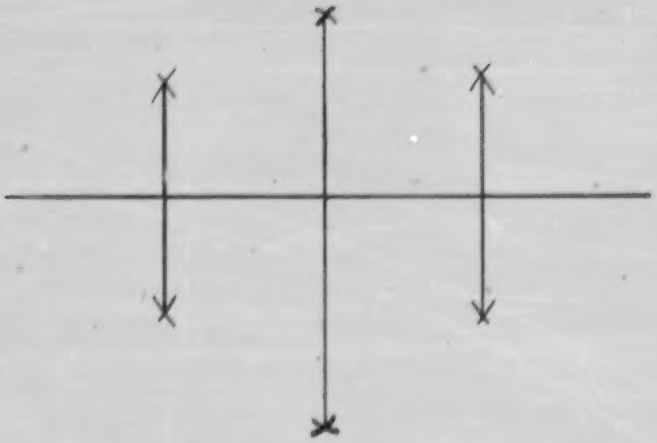


7. (a)

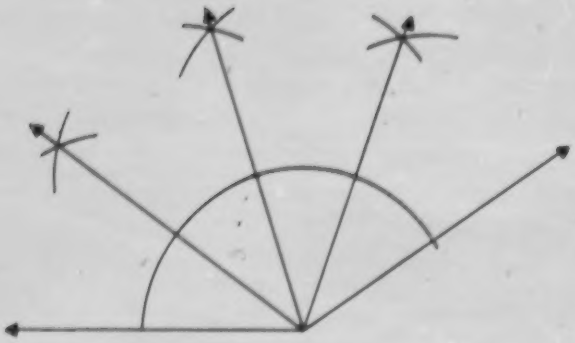


(b) The three bisectors meet in a point.

8.



9.



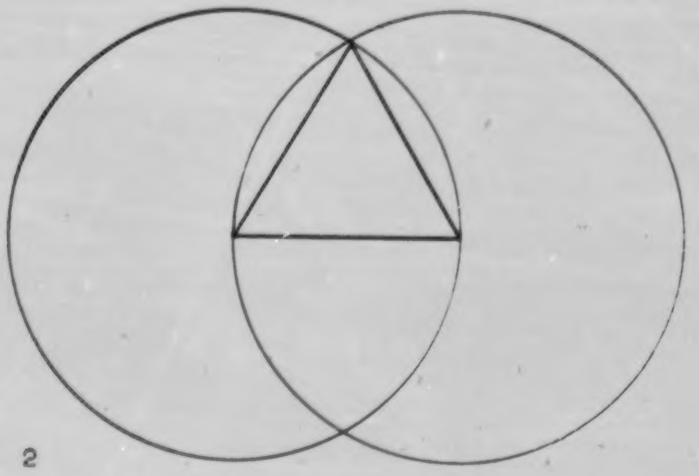
Without putting too much stress on the topic an attempt is made to let the student develop constructions for triangles to match the three basic conditions of congruency. No mention of congruence is made at this time, but, as a prelude to congruence, the student is asked whether all triangles with three measurements the same will look alike. Students probably will need help to see that triangles that are rotated or are mirror images of each other are alike.

More will be done with these constructions when the topic of congruence is studied in Section 4-4. It is hoped that the better students will be able to develop these concepts for themselves with the hints given with these problems. If it proves too difficult, the teacher may want to enlarge upon the problems.

The concept of concurrent lines is used in connection with the interesting sets of concurrent lines found in triangles. The concept of concurrent lines is simple and should cause no problem; the better students may be sufficiently intrigued to do some independent work with this.

Answers to Exercises 4-2b

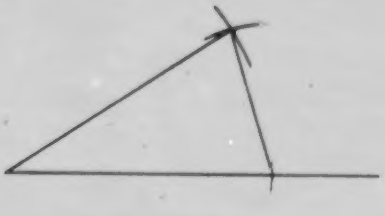
1. (a), (b), and (d)



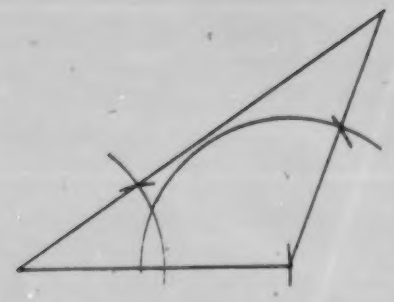
(c) 2

(f) equilateral

2.

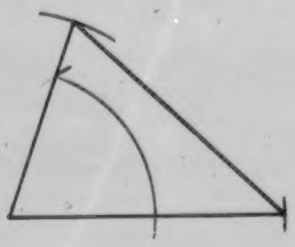


3. (a)



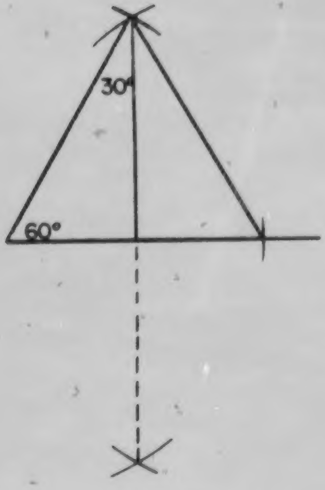
(b) Yes; but one might be a mirror image of the other.

4. (a)

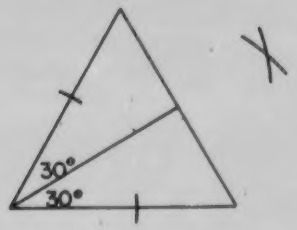


(b) Yes; but one might be a mirror image of the other.

5.

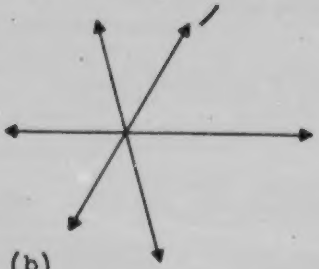


by Construction 2

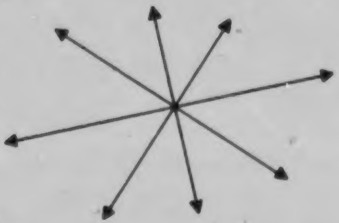


by Construction 4

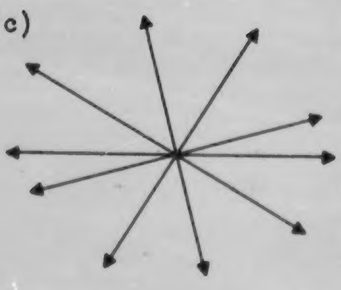
6. (a)



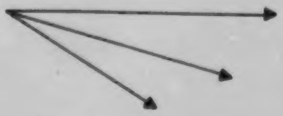
(b)



(c)

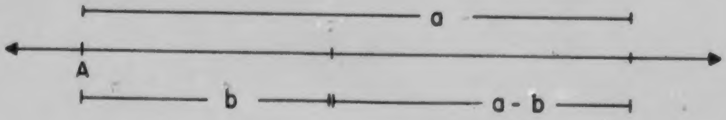


7. (a)

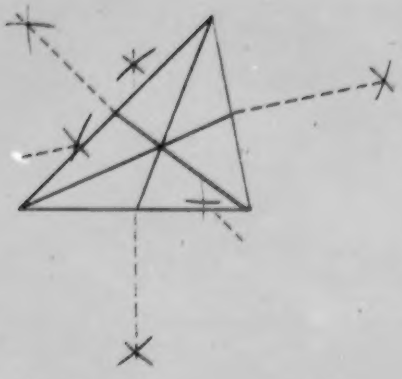


(b) 3; the two small angles and the large angle. Using exterior angles or equally spaced rays 6 angles are possible.

8.



9. (a)



(b) yes

Methods for inscribing triangles, squares, hexagons, and octagons are developed in Exercises 4-2c. With these basic constructions and ideas, students can copy geometric designs and can create their own. This is a topic in mathematics where many students who have not been outstanding previously, have an opportunity to shine.

Students who have had SMSG work in seventh grade are familiar with the term circle as a set of points that are equal in distance from a central point and should be reminded of the definition. Since this definition eliminates the interior (or area) of the circle as part of the circle, students who have had other work should learn to consider the term "circle" to be the same as "the circumference of the circle." After the basic processes are learned, most eighth-grade students enjoy creating their own designs. This need not take much class time since most of this can be done outside of class.

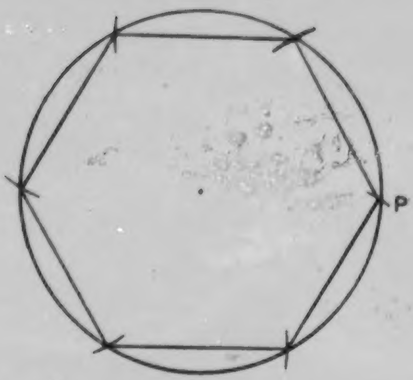
Answers to Exercises 4-2c

1. (a) 6

   (c) Hexagon

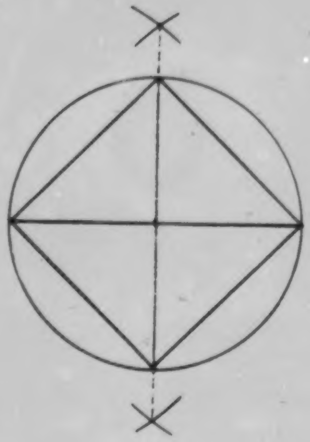
   (d) Connect every other intersection.

   (e) Connect every other intersection and then connect those omitted at first.

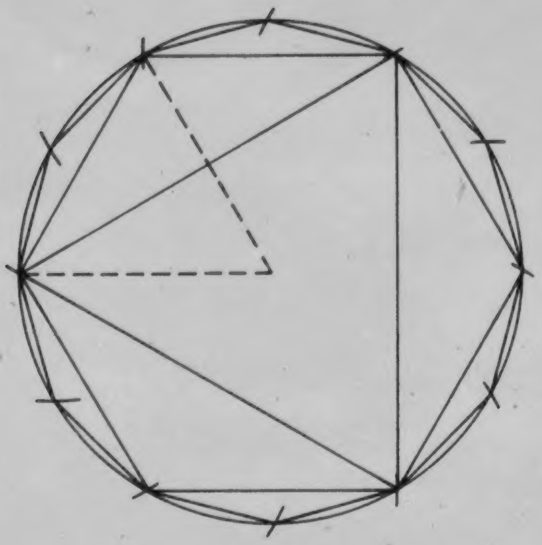


2. (a) Square

   (b) Bisect each angle and extend the rays to the circle forming 8 diameters. Connect the endpoints in order. An alternate method is to bisect each side and draw the bisectors so that they intersect the circle. Again connect the 8 points of intersection in order.



3.



4-3. Symmetry

This section introduces the concept of symmetry of a plane figure with respect to a line in order to prepare the pupil for ideas about congruent triangles. Symmetry with respect to a point and symmetry with respect to a plane are relegated to roles of less importance because, in the subsequent development, they are used only in Section 4-9 where plane symmetry is brought in to assist the student in visualizing three-dimensional figures.

The pupil is to obtain his basic ideas of symmetry from class discussion of figures obtained by cutting from folded sheets of paper. The definition of symmetry with respect to a line is delayed in order to encourage the use of intuition, rather than rule. This section is in no sense intended to be a complete or formal treatment of the interesting and fascinating topic of symmetry.

Answers to Class Exercises 4-3

1. (a) Triangular

   (c) yes

   (d) 1, 3, 0

2. (b) Parallelogram

   (c) Yes. The halves fit. Yes. 2.

3. Yes. The lines bisecting the angles at the center are also axes of symmetry .6.

4. Yes. Every diameter is on an axis of symmetry. The number of axes of symmetry is greater than any number you might name. (If students are aware of the word "infinite," it is correct to say that there are an infinite number of axes of symmetry.)

5. Yes. The line perpendicular to $\overline{AB}$ at its mid-point is an axis of symmetry.

   2.

   $\overline{AB}$ is called the major axis because $\overline{AB}$ is longer than the segment cut by the ellipse on the other axis of symmetry. The minor axis is perpendicular to $\overline{AB}$ at its mid-point.

Answers to Exercises 4-3

1. 2. For a square, 4.

2. 3.

3. 4.

4. (a) 1

   (b) 1

   (c) 2

   (d) 2

   (e) 4

   (f) 2

   (g) 0

   (h) 3

5. Yes. Yes.
6. The two folds.
7. A plane figure is symmetrical with respect to a point if for every line through the point the only intersections with the figure are pairs of points such that the two members of a pair are equally distant from the point of symmetry.

   c, d, e, f.
8. Sphere, cone, prisms, pyramid, many fruits when cut in half such as apples, bananas and grapefruit.

4-4. Congruent Triangles

The background for the section is built in the section on constructions. Notice that the word "congruent" is used to describe segments and angles formerly described as equal.

In this section correspondence is really a matter of choice. You are given, say, two triangles ABC and RST. A correspondence between the vertices can be set up in any one of six ways. Once such a correspondence has been set up, we decide that corresponding sides shall be opposite corresponding angles. With this understanding, a correspondence of vertices determines a correspondence of sides. Another way to set up a correspondence would be to start with the sides, and have it induce a correspondence of vertices. Usually we choose to set up a correspondence so that congruent angles correspond or congruent sides correspond.

In an informal treatment of congruence, such as the one presented in this chapter, we have said that two sets of points are congruent if they have the "same size and shape." In traditional terminology, this is interpreted as meaning "if either figure (set of points) can be superimposed on the other." The process of superposition gets us involved with considerations of "moving objects around," and, from some points of view, the motion involved is irrelevant to the idea of congruence. Also while we shall be

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primarily concerned with congruence between sets of points in a plane, the definition we use is applicable to sets of points in space. The idea of superimposing one billiard ball on another doesn't make much sense. Yet billiard balls are "congruent."

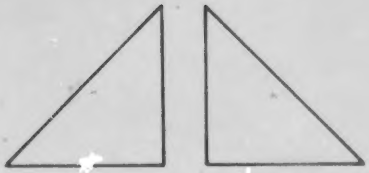
Suppose ΔPRQ can be superimposed on ΔABC with R falling on B, P on A and Q on C. Then there exists a one-to-one correspondence between ΔPRQ and ΔABC, each point of ΔPRQ corresponding to that point of ΔABC which it "covers" when ΔPQR is superimposed on ΔABC. For example, the point X would correspond to the point X' under this correspondence. But it is not enough simply to say that there exists a one-to-one correspondence between ΔPRQ and ΔABC. Something else is also involved in the notion of congruence. Distances must be preserved. Suppose ΔPRQ is superimposed on ΔABC, then for any two points of ΔPRQ, the distance between them must be the same as the distance between the two points of ΔABC which they cover, i.e., between the two points of ΔABC to which they correspond under the one-to-one correspondence. As examples, the distance between R and X must be the same as the distance between B and X' (in other words, $RX \cong BX'$).

These considerations lead to a more formal definition:

Definition. Two sets of points in a plane are said to be congruent provided that there is a one-to-one correspondence between them which preserves distance.

While this definition is not included in the student text there is nothing in the text which is contradictory to this definition. When the student studies geometry in high school he should be well able to appreciate a more precise treatment of congruence.

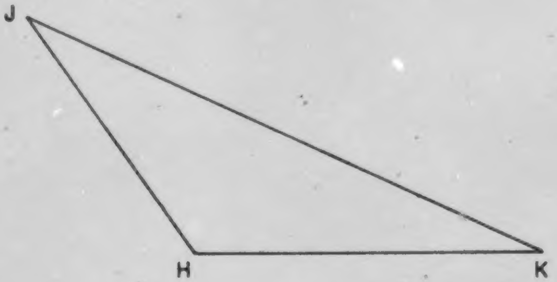
If a question arises about two triangles (as shown) not being congruent, because they are not the same "shape," it should be pointed out that we consider them to be the same shape because one could be superimposed on the other by paper folding, and also that the correspondence between congruent angles and sides can be readily established. See Problem 18 of this section.



It is hoped that with work on triangles in Section 2, especially in Class Exercises 4-2a, students can be left an opportunity for some "discovery" in reaching the conclusions stated in Properties S.S.S., S.A.S., and A.S.A. This kind of experience should assist the student in applying the properties in problem situations. Problems 3 through 17 are included to provide for practice in recognizing pairs of congruent triangles, using the 3 properties.

Answers to Exercises 4-4

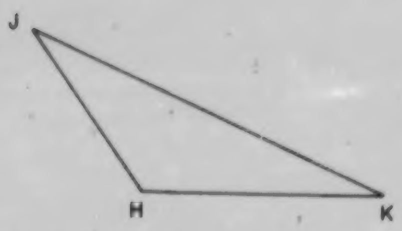
1. (a)



(b) and (c) are the same as (a).

(d) Yes. Property S.S.S., Property A.S.A., and Property S.A.S.

2.



(a) If two angles of a triangle are known, the third is determined.

(b) No.

(c) Because two triangles with congruent angles (in pairs) do not necessarily have congruent sides.

3. Congruent. Property S.S.S.

4. Congruent. Property S.A.S.

5. Congruent. Property S.S.S. (both are equilateral)

6. Not necessarily congruent. Congruent sides not between corresponding angles.

7. Congruent. Property A.S.A.

8. Congruent. Property S.A.S.

9. Congruent. Property A.S.A.

10. Congruent. Property S.S.S.

11. Not necessarily congruent.

12. Congruent. Property A.S.A.

13. Congruent. Property A.S.A.

14. Congruent. Property S.A.S.

15. Congruent. Property S.S.S.

16. Not necessarily congruent.

17. Not congruent; triangle ABC is equilateral, PQR cannot be equilateral.

18. You cannot slide one triangle along the paper so that it will coincide with the other. It would have to be turned over. The paper might be folded along the axis or line of symmetry.

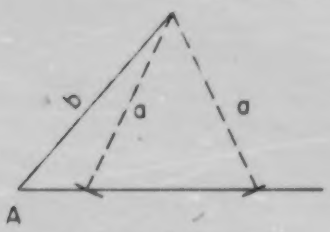
19. BRAINBUSTER. The 6 cases to be considered are:

1. side, side, side — Property S.S.S.
2. side, included angle, side — Property S.A.S.
3. side, side, angle opposite one of the sides
4. angle, angle, angle
5. angle, included side, angle — Property A.S.A.
6. angle, angle, side opposite one of the angles

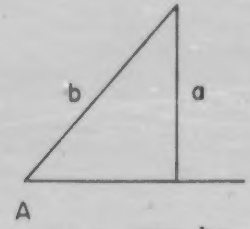
Cases 1, 2, 4, 5 have been considered.

Case 3. (so-called ambiguous case in trigonometry)

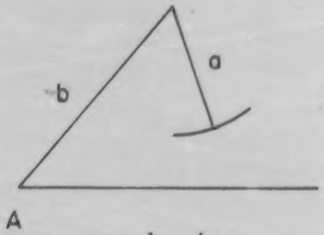
Given sides a and b, and angle A.



a may be long enough to form two Δ's with b, ∠A, a.



a may be just long enough to form a right triangle.



a may be too short to form any triangle.

Case 6. If two angles are given, the third angle may be determined, and, hence, Case 6 reduces to Case 5.

It may be observed that among the 6 parts of a triangle, there are 20 groups of 3 parts each. However, for example, a, C, b; a, B, c; and b, A, c all reduce to Case 2.

4-5. Showing Two Triangles to be Congruent

Because the idea of proof is new to students at this age, a separate section has been prepared in which the student is asked to provide an informal "proof." The authors deliberately avoid using the word "proof" in the text except in the title of Section 4-7. No particular form is suggested for proofs, so long as the student includes all steps and gives his reasons carefully.

In the first proof given in this section, emphasize that the isosceles triangle property has been shown or proved to be true, if the Property S.A.S. is true. Property S.A.S. is, of course, based on experience and not on proof. If the student says why prove this new property when we could base it on experience, also, point out the advantage of proof based on as few properties as possible for which we depend on experience only.

One of the difficulties in early experiences with proof is for the student to see why a proof is needed. The construction of perpendiculars was delayed until this section because it should help the student appreciate the need to show or prove that a construction really does do what is claimed for it.

In showing that two lines are perpendicular, it is necessary to think of the sum of the measures of two congruent angles where the sum is 180. In these texts for grades 7 and 8 we do not define a straight angle, as is done traditionally, nor do we talk about an angle of measure 180. In Euclidean geometry, an angle is simply a geometric figure, that is, a set of points. Thus, we define an angle by saying: "An angle is a set of points consisting of two rays with an endpoint in common and not both on the same straight line." The reason for excluding "straight angles" is that on a "straight angle" it is impossible to tell where the vertex is supposed to be; any point of a line will do, as the "vertex" of the corresponding "straight angle." It was no doubt for this reason that Euclid did not use them.

In analytic geometry we deal with sensed angles, or ordered angles. These are ordered pairs of rays, with the same endpoint. The first ray in the pair is called the initial side, and the second ray in the pair is called the terminal side. The measure of a sensed angle depends on the order in which the sides are taken. For example, if a sensed angle has measure 270°, then when we reverse the order of the sides we get an angle of measure -270° or 90°. The sides of a sensed angle may be collinear; in this case we get a straight angle. Also, the sides of a sensed angle may be identical; in this case we get an angle of 0°.

The idea of a sensed angle is, of course, more difficult. The definition we used is sufficient for the purposes of elementary geometry. For example, the angles of a triangle never have measure 0 or 180, and there is no natural way to decide on an order in which the sides ought to be taken. Therefore, in an introductory treatment, we defined angles in the simplest way possible, postponing the more difficult idea until such time as it would be needed.

Be sure to assign and discuss Problem 5 of Exercises 4-5a, since this same figure will be used again in Chapter 9.

Answers to Exercises 4-5a

1. (a) $\overline{AB} \cong \overline{BC}$, $\overline{AD} \cong \overline{CD}$.
   $\overline{BD} \cong \overline{BD}$.
   (b) Yes. Property S.S.S.
   (c) Yes. If two triangles are congruent, then each pair of corresponding angles is congruent.
2. (a) $\overline{EF} \cong \overline{HJ}$, $\overline{FG} \cong \overline{JK}$, $\overline{EG} \cong \overline{HK}$.
   (b) Yes. Property S.S.S.
   (c) Yes. Property: If two triangles are congruent, then each pair of corresponding angles is congruent.

3. (a) Yes. $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$.
   (b) No. There is not a pair of sides equal.
   (c) Triangles would be congruent. Property A.S.A.

4. (a) Yes. Property S.A.S.
   (b) Same length as $\overline{QR}$ which can be measured.

5. (a) Yes.
   (b) Property S.A.S. or Property A.S.A.

6. (a) $\angle 1 \cong \angle 2$ because they are corresponding angles.
   (b) Yes, they are vertical angles.
   (c) Since $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$, $\angle 1 \cong \angle 3$.

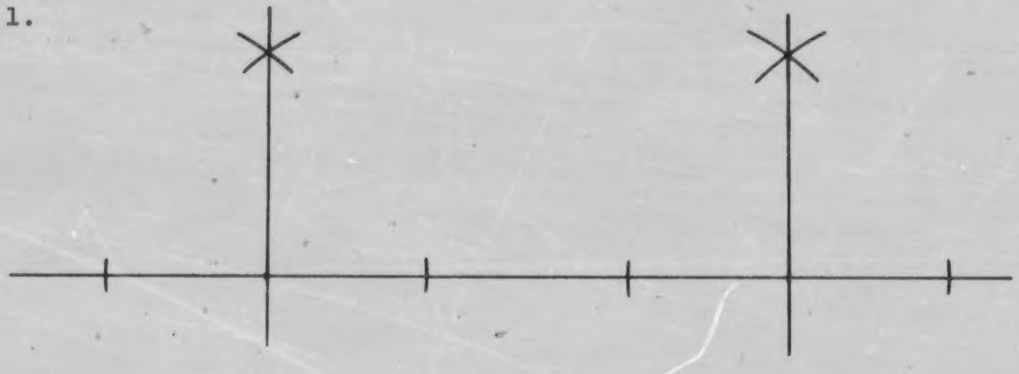
7. (a) Yes, corresponding angles.
   (b) Vertical angles. Yes, they are congruent.
   (c) They are congruent.
   (d) $\angle 6 \cong \angle 7$ because they are vertical angles.
       $\angle 4 \cong \angle 5$ in a manner similar to $\angle 1$ and $\angle 3$.
   (e) No, one side is also needed.
   (f) Yes, $\overline{AB} \cong \overline{CD}$ because they are opposite sides of a parallelogram.
   (g) $\angle 1 \cong \angle 3$
       $\angle 4 \cong \angle 5$
       $\overline{AB} \cong \overline{CD}$
       Therefore, $\Delta ABE \cong \Delta CDE$ because of Property A.S.A.
       Hence, $\overline{BE} \cong \overline{DE}$ and $\overline{AE} \cong \overline{CE}$ and the diagonals bisect each other.

8. Assume $\angle A \cong \angle B$.
In triangles ABC and BAC
$\overline{AB} \cong \overline{BA}$
$\angle A \cong \angle B$
$\angle B \cong \angle A$
$\triangle ABC \cong \triangle BAC$ Property A.S.A.
Then $\overline{AC} = \overline{BC}$ corresponding sides of congruent triangles

The construction of a perpendicular to a line from a point not on the line is not given as a separate construction. The teacher may wish to let the students discover how to modify appropriately the construction of a perpendicular to a line at a point on the line, or he may teach the construction of a perpendicular from a point as a separate construction before assigning the problems. In either case the similarity of the constructions should be made clear.

Answers to Exercises 4-5b

1.



2. See the figure for Problem 2 in the student text.

3. (a)

$\overline{AP} \cong \overline{BP}$	by construction
$\overline{AQ} \cong \overline{BQ}$	by construction
$\overline{PQ} \cong \overline{PQ}$	common side
$\triangle APQ \cong \triangle BPQ$	Property S.S.S.

(b) Corresponding angles of congruent triangles.

(c) $\angle PAQ \cong \angle PBQ$
$\angle AQP \cong \angle BQP$

4.

$\overline{AP} \cong \overline{BP}$	by construction
$\angle 1 \cong \angle 2$	see Problem 3(b)
$\overline{PC} \cong \overline{PC}$	common side
$\triangle APC \cong \triangle BPC$	Property S.A.S.
$\angle ACP \cong \angle BCP$	corresponding angles of congruent triangles

The sum of the measures of $\angle ACP$ and $\angle BCP$ is 180.
The measure of $\angle ACP$ is 90.
$\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$

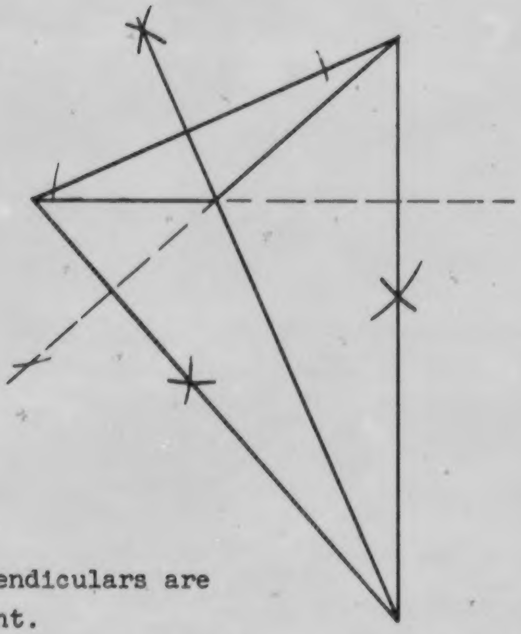
5. This is essentially the same problem as Problems 3 and 4, except different letters are used. After it is shown that
$\triangle ECO \cong \triangle EDO$,
then $\overline{CO} \cong \overline{DO}$ corresponding sides of congruent triangles

6. (a) Triangle ECF $\cong$ triangle EDF
Triangle COE $\cong$ triangle DOE
Triangle COF $\cong$ triangle DOF

(b) $\overline{CE} \cong \overline{DE}$
$\overline{CF} \cong \overline{FD}$
$\overline{EF} \cong \overline{EF}$
$\overline{CO} \cong \overline{OD}$
$\overline{EO} \cong \overline{EO}$
$\overline{OF} \cong \overline{OF}$

(c) $\angle CEO \cong \angle DEO$
$\angle ECO \cong \angle EDO$
$\angle 1 \cong \angle 2$
$\angle CFO \cong \angle DFO$
$\angle FCO \cong \angle FDO$
$\angle COF \cong \angle DOF$
$\angle ECF \cong \angle EDF$

7.



The perpendiculars are concurrent.

8. In bisecting a line segment, as a first step one locates a point not on the segment or the segment extended which is the same distance from each endpoint of the segment.
In constructing a perpendicular, as a first step one locates two points on the line which are the same distance from the point through which the perpendicular is to be constructed (whether the point is on the line or not).
The second steps in each construction are the same, and the drawing of the line, which is the bisector or the perpendicular, is essentially the same in both cases.

4-6. The Right Triangle

This section develops the concept of the Pythagorean Property intuitively. Those classes that are capable of a more rigorous proof of this property of the right triangle will find a development given in the next section.

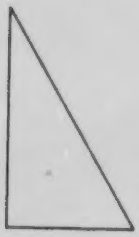
Two special right triangles are used for the intuitive development. The 3-4-5 triangle is used both for understanding and also to give a historical background. An isosceles right triangle strengthens the understanding of the Pythagorean relationship. After the student has made observations using these two triangles, the Pythagorean Property is stated with a note that the student has seen it in only two special cases but that it is true for all right triangles.

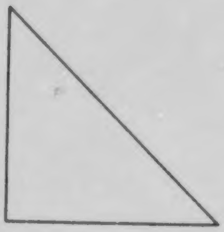
The necessity for square roots requires the introduction of both the concept and the symbol at this time. Only the positive square root and symbol are used but no point is made of the fact. Some students may be curious about the use of the word "positive"; the teacher could discuss the negative root but suggest that at this time they will use only the positive root and should leave a study of negative roots for later work.

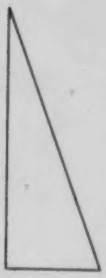
A suggestion is made for estimating the square root of a number but full development is left for the chapter on real numbers.

Answers to Exercises 4-6a

1.







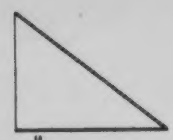
2. (a) $5^2 = 4^2 + 3^2$
$25 = 16 + 9$
$25 = 25$

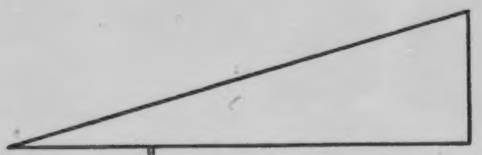
(b) $13^2 = 5^2 + 12^2$
$169 = 25 + 144$
$169 = 169$

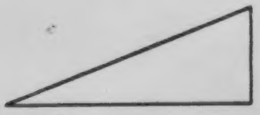
(c) $25^2 = 7^2 + 24^2$
$625 = 49 + 576$
$625 = 625$

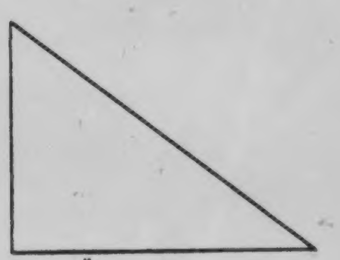
(d) $20^2 = 16^2 + 12^2$
$400 = 256 + 144$
$400 = 400$

3.

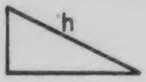

$\frac{1}{4}$" = 1 Unit

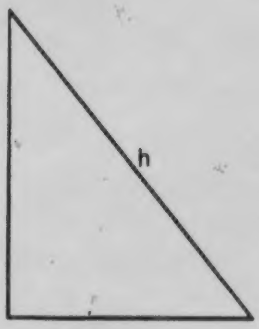

$\frac{1}{8}$" = 1 Unit

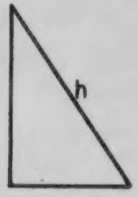

$\frac{1}{8}$" = 1 Unit


$\frac{1}{8}$" = 1 Unit

4.







5. (a) Area is 5 sq. cm.

(b) Area is 41 sq. cm.

(c) Area is 13 sq. cm.

Answers to Exercises 4-6b

1. (a) $\sqrt{5} \approx 2.236$
   (b) $\sqrt{41} \approx 6.403$
   (c) $\sqrt{13} \approx 3.606$
   (d) $\sqrt{92} \approx 9.592$
   (e) $\sqrt{676} = 26$
   (f) $\sqrt{5625} = 75$

2. (a) App. 2.236"
   (b) App. 6.403'
   (c) App. 3.606"
   (d) App. 7.810 yd.
   (e) App. 9.487'
   (f) App. 3.162 units

3. (a) 12'
   (b) 10'
   (c) 36'

4. 17 ft.

5. $16\frac{1}{2}$ ft.

6. 14 ft.

7. App. 6.4 ft.

8. 58 ft.

9. App. 85 ft.

10. The number of units in each case is $\sqrt{2}$ which is approximately 1.4.

11. The number of units is $\sqrt{3}$ which is approximately 1.7.

4-7. One Proof of the Pythagorean Property

The word "proof" is used in the title of this section but avoided in the rest of the section. Students are asked to "show" that something is true. While the development leads to a logical proof, it is given in an informal manner more suitable for students of this age. Students who follow this work will probably need considerable help from the teacher; a satisfying sense of achievement will reward those students capable of understanding this approach.

The proof that is presented is, of course, not the one attributed to the Pythagoreans, since that proof is much too difficult at this time. The dissection of congruent squares in two different ways is the basis for the proof given. It would be advisable for the teacher to be thoroughly familiar with this development before teaching this to a class or allowing a few students to work on it alone or in a small group.

4-8. Quadrilaterals

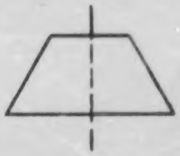
This section is used as a summary and review section. If time is short the section could be omitted. Most students, however, will find the section to be rather easy. Concepts introduced earlier in the chapter are used in various relationships found in quadrilaterals. As a reminder, the various quadrilaterals are named and defined.

Answers to Exercises 4-8

1. (c) Rectangle and (d) Square

2. (a) 2 (b) 4

3. Yes



4. Yes, any rectangle or square.

5. (a) 360°. All quadrilaterals can be separated into two triangles by a diagonal. The sum of the measures of the angles in a triangle is 180, so, for two triangles, it must be 360.

   (b) Parallelograms and the subsets, rectangles and squares.

6. 80 inches

7. 25 feet

8. Figure A

   (a) 8

   (b) No.

   Figure B

   (a) 8

   (b) Yes. ΔABC, ΔCDA
   ΔABD, ΔBCD
   ΔAOD, ΔCOB
   ΔAOB, ΔCOD

   Figure C

   (a) 8

   (b) Yes. ΔABC, ΔADC
   ΔABD, ΔBCD
   ΔAOB, ΔBOC, ΔCOD, ΔDOA

   Figure D

   (a) 8

   (b) No.

Figure E

(a) 8

(b) Yes. $\triangle ABC$, $\triangle CDA$
$\triangle ABD$, $\triangle BCD$
$\triangle AOB$, $\triangle COD$
$\triangle AOD$, $\triangle COB$

Figure F

(a) 8

(b) Yes. Same $\triangle$s as in Figure C.

(c) Proof of congruence can be given for two sets of congruent triangles. One or the other applies in all cases of congruence.

1. Prove $\triangle ABC \cong \triangle CDA$
$\overline{AB} \cong \overline{CD}$ (opposite sides of a parallelogram)
$\overline{BC} \cong \overline{DA}$ (opposite sides of a parallelogram)
$\overline{AC} \cong \overline{AC}$ Same segment
$\therefore$ $\triangle ABC \cong \triangle CDA$ (because of Property S.S.S.)

2. Prove $\triangle ABO \cong \triangle DCO$
$\overline{AB} \cong \overline{CD}$ (opposite sides of a parallelogram)
$\angle ABO \cong \angle CDO$; (alternate interior angles formed by parallel lines and a transversal. See Part (c) Problem 8.)
$\angle BAO \cong \angle DCO$
$\therefore$ $\triangle ABO \cong \triangle DCO$ (because of Property A.S.A.)

If students remember that the diagonals of a parallelogram bisect each other, these triangles can be proved congruent by use of Property S.S.S. also.

4-9. Solids

No effort is made to describe these solids completely in this chapter, since they will be studied in more detail later. Enough of a description is given, however, to enable the pupils to proceed with making the drawings.

Perhaps the most difficult concept is that of projection. Pupils who have had an art course may understand this concept. All members of the class can be helped through a proper discussion of this topic.

Drawing pictures of 3-dimensional objects on 2-dimensional space demands some distortion. Reality can be gained by use of perspective or projection. The drawings used in this section make use of diametric projection. Students may be helped if this is compared to flat maps which use some projection, usually the Mercator. The word perspective is used loosely to indicate a distortion that appears to give depth. Getting the proper "perspective" is the crux of this section. Of course practice is essential in obtaining the desired results.

Problems 7, 8 and 9 require some knowledge of symmetry with respect to a plane. This concept was introduced in Exercises 4-3, Problem 8. No attempt is made to define symmetry with respect to a plane, but it should be easy to develop an intuitive notion of plane symmetry by asking students to think of a chalk-box divided into two equal parts, a chair cut in a certain way down the middle, or the two halves of an English-walnut shell.

Answers to Questions in Text

(I) (2) Triangular Right Prisms: (f) 5 faces
(g) two are triangles and three are rectangles

(3) Hexagonal Right Prisms: (d) 8 faces
(e) 2 faces are hexagonal and 6 are rectangular.

(II) Pyramids: (e) 4 faces. The base is considered a face.

(III) Intersecting Planes: (d) Parallelogram

Answers to Exercises 4-9

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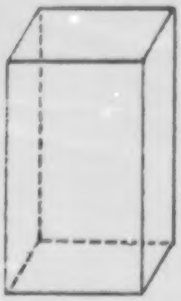

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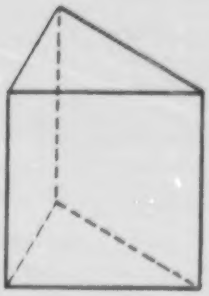

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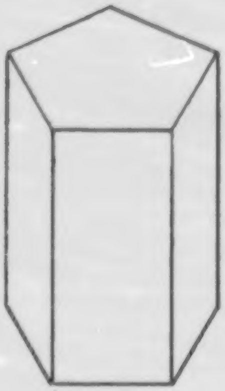

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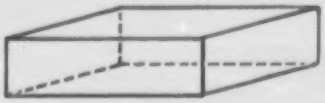

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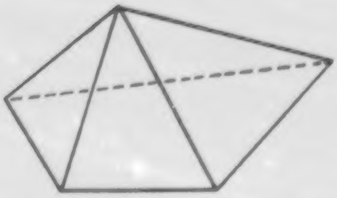

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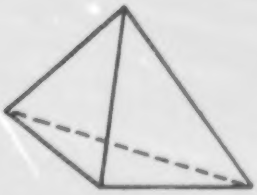


7. (a) 3 pairs of parallel faces

*(b) 3 planes each of which divides the pairs of the parallel rectangular faces into two congruent rectangles.

8. (a) Parallel faces are congruent triangles and the other 3 faces are congruent rectangles.

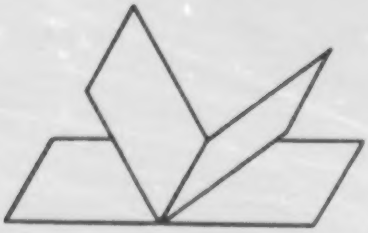
*(b) 3 planes, each of which divides each of the parallel triangular faces into a pair of congruent triangles and a fourth plane which divides each of rectangular faces into pairs of congruent rectangles.

*9. (a) Parallel faces are congruent triangles. (No congruent rectangles.)

(b) 1 plane parallel to the parallel faces.

10. Not necessarily.

11.



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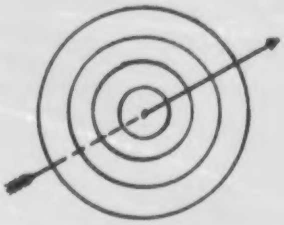


Sample Questions

Select from this list. There are far too many questions for one test. Particularly, on the content of this chapter, the questions should be carefully chosen and the test should not be too long.

True-False

T 1. Congruent triangles have the same area.

T 2. The two parts of a figure that are symmetrical with respect to a line are congruent.

F 3. A trapezoid always has an axis of symmetry.

F 4. The capital letter C has two axes of symmetry.

T 5. Every circle is symmetrical with respect to all its diameters and to its center.

F 6. Two triangles are congruent if three angles of one are respectively congruent to the three angles of the other.

T 7. Two rectangles are congruent if the bases are congruent and the altitudes are congruent.

T 8. Two triangles are congruent if two sides and the included angle of one are congruent to two sides and the included angle of the other.

F 9. Two triangles are congruent if corresponding angles are congruent.

F 10. Right triangles have two right angles.

F 11. The hypotenuse of a right triangle is on one ray that forms the right angle.

T 12. The length of the hypotenuse is less than the sum of the lengths of the other two sides of a right triangle.

T 13. An isosceles triangle may also be a right triangle.

F 14. An equilateral triangle may also be a right triangle.

T 15. A triangle whose sides measure 80, 84 and 116 is a right triangle.

Multiple Choice

E 1. A trapezoid must have

- A. Two pairs of parallel sides.
- B. At least one pair of congruent sides.
- C. Diagonals that always bisect each other.
- D. One right angle.
- E. None of these.

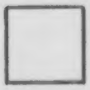
C 2. Rectangles are included in:

- A. The set of squares.
- B. The set of trapezoids.
- C. The set of parallelograms.
- D. The set of pentagons
- E. None of these.

D 3. The diagonals of some figures bisect each other. In which of these groups is this true for all figures of the group?

- A. Quadrilaterals, parallelograms and squares.
- B. Trapezoids, parallelograms and rectangles.
- C. Quadrilaterals, parallelograms and rectangles.
- D. Parallelograms, rectangles and squares.
- E. None of these.

A 4. All of these figures have at least one axis of symmetry.

1.


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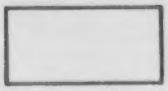

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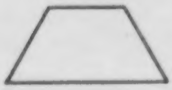

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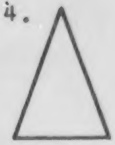

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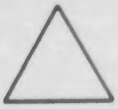


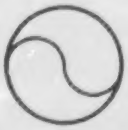
Select the group in which all figures have more than one axis of symmetry.

A. 1, 2, and 5.
B. 2, 3, and 4.
C. 2, 3, and 5.
D. 1, 2, and 3.
E. All of these.

Completion

6 5. A regular hexagon has ______ lines of symmetry.

point
line
6. This figure represents symmetry with respect to a __________ but not to a ________.



corre-sponding parts 7. If two figures are congruent, then __________ __________ are congruent.

8. The Pythagorean Property states that the area of

____

____

(ANSWER--The area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.)

7 9. The hypotenuse of a right triangle is 25 units in length, and one side is 24 units in length. The other side is ________ units in length.

Drawings and Constructions

Answers for constructions and problems (10-30) are at the end.

In test items 10 through 17, use a ruler and protractor to draw the figures: Items 14 through 17 are solids.

10. Draw a triangle that has a segment 3 inches long determined by angles of 35° and 65°.

11. Draw a triangle that has an angle of 100°. The segments on each side of this angle are $2\frac{1}{2}$ inches and $3\frac{1}{4}$ inches long.

12. Draw an isosceles triangle whose base is $1\frac{1}{2}$ inches and whose equal angles are each 40°.

13. Draw a regular pentagon with a side $\frac{1}{2}$ inch long. Each angle in a pentagon is 108°.

14. Draw a cube.

15. Draw a triangular prism.

16. Indian tepees may have the shape of pyramids. Draw a tepee of this type.

17. A monument has a square prism for the base topped by a square pyramid. Draw a picture of such a monument.

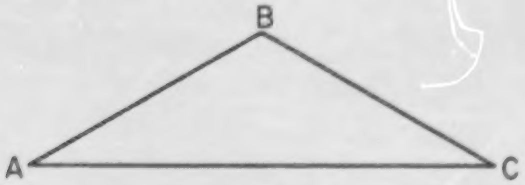
In test items 18 through 23, use only straightedge and compass to make the constructions.

18. Construct a square with each side 2 inches long. (Use a ruler to measure this segment but use a compass to copy the sides from the segment.)

19. Construct an equilateral triangle.

20. Construct a triangle with sides 2 inches, 3 inches, and 4 inches in length.

21. Construct a triangle EFG congruent to triangle ABC.



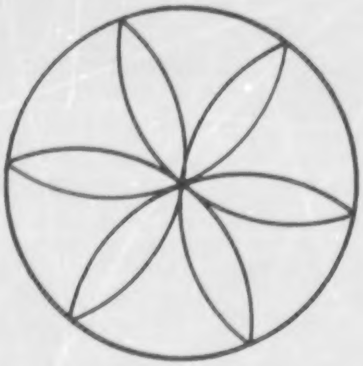
List the pairs of congruent angles and pairs of congruent sides.

22. Why is it impossible to construct a triangle with sides of 1 inch, 2 inches and 3 inches? Show this by a construction.

23. Construct a triangle with angles of 30°, 60°, and 90°.

Problems

24. Draw all axes of symmetry in this figure.



25. Triangle ABC is an isosceles triangle. $\overline{AB} \cong \overline{AC}$. M is the mid-point of $\overline{BC}$.

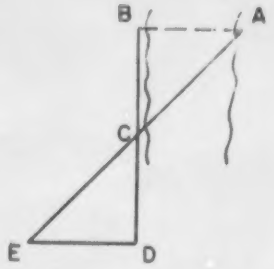
    (a) Show that $\triangle BAM \cong \triangle CAM$.

    (b) Show that the angles opposite the congruent sides are congruent, that is $\angle B \cong \angle C$.

26. The length of the hypotenuse of a right triangle is 41 inches. One side has a length of 9 inches. Find the third side.

27. A rectangular playing area is 80 feet long and 60 feet wide. What is the length from one corner to the opposite corner?

28. In order to find the width of a creek, Bob located points A and B directly opposite each other. From point B he marked $\overline{BD}$ at right angles to $\overline{AB}$. Point C is the mid-point. Bob then marked $\overline{DE}$ so that $\overline{DE}$ is perpendicular to $\overline{BD}$ and so that A, C and E all lie on $\overleftrightarrow{AE}$.

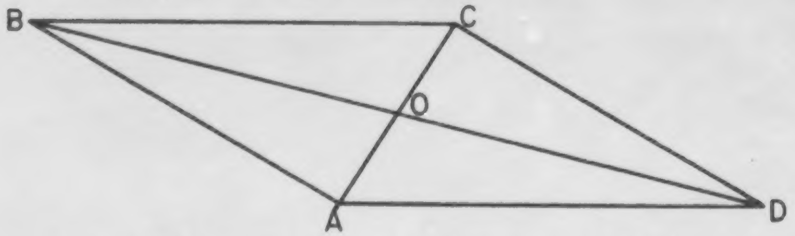


    (a) If the length of $\overline{CD}$ is 15 ft., of $\overline{DE}$ is 36 ft., and of $\overline{EC}$ is 39 ft., how wide is the creek?

    (b) This method of measure is based upon the congruence of $\triangle ABC$ and $\triangle EDC$. Show that these triangles are congruent.

29. In a parallelogram ABCD, $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$. Show that $\angle ABD \cong \angle CBD$. Sketch a figure and label it to help you do this.

30. In the parallelogram ABCD:



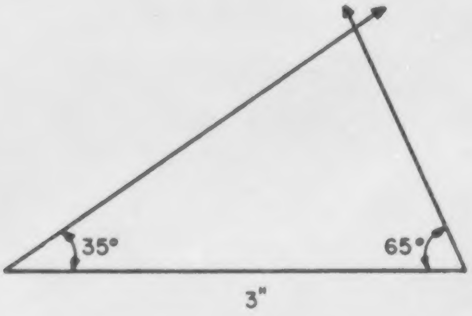
(a) List as many pairs of congruent triangles as you can.

(b) List the congruence of each pair of corresponding angles in triangles AOB and COD.

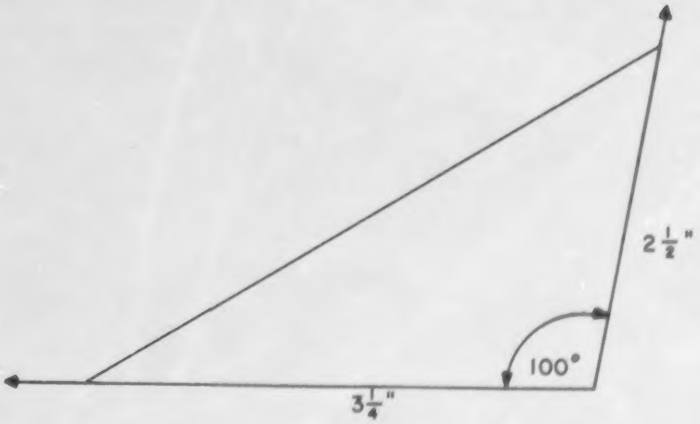
(c) Is AD ≅ BC? Why?

Answers to Sample Questions, 10 through 30

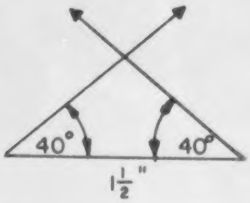
10.



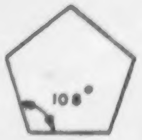
11.


$2\frac{1}{2}$"
100°
$3\frac{1}{4}$"

12.


40°
40°
$1\frac{1}{2}$"

13.



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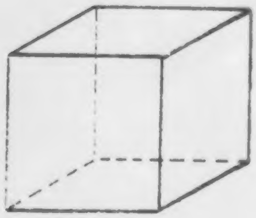

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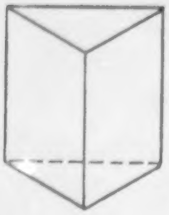

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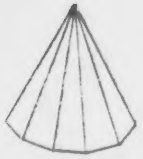

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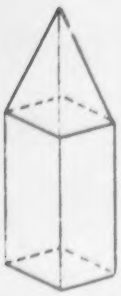

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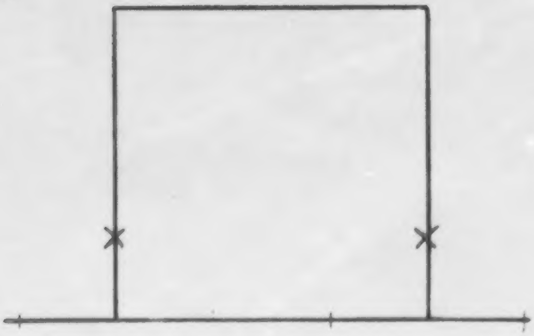

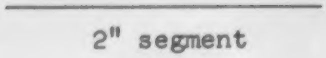

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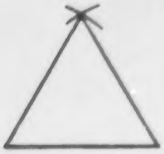

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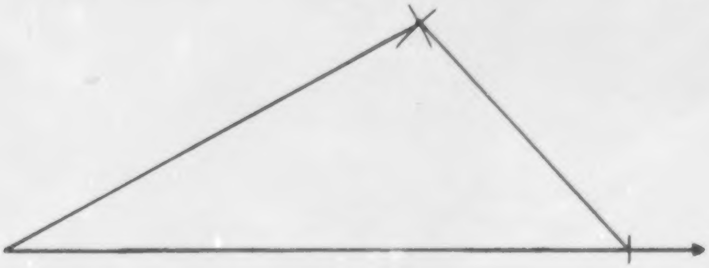

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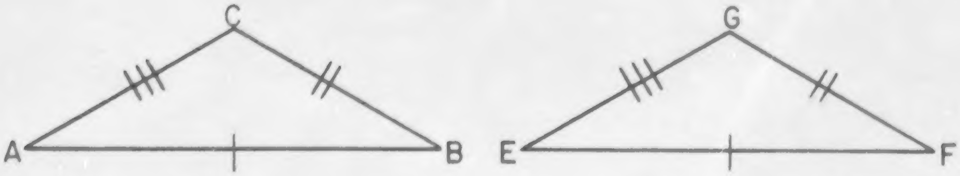
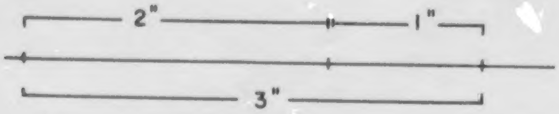


$\overline{AC} \cong \overline{EG}$
$\overline{CB} \cong \overline{GF}$
$\overline{BA} \cong \overline{FE}$

$\angle A \cong \angle E$
$\angle B \cong \angle F$
$\angle C \cong \angle G$

22. $1 + 2 = 3$. The sum of the lengths of the two shorter segments must be greater than the length of the largest segment.



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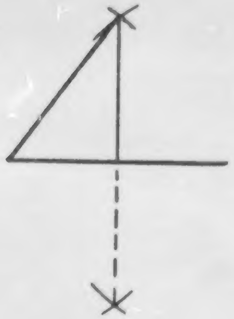
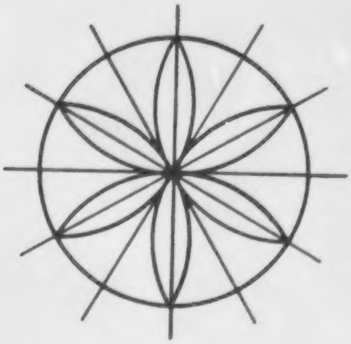


(There are other ways that this can be done. This is the easiest.)

24.



25.

(a)	$\overline{AB} \cong \overline{AC}$	given
	$\overline{BM} \cong \overline{CM}$	M is mid-point of $\overline{BC}$
	$\overline{MA} \cong \overline{MA}$	common side
	$\triangle ABM \cong \triangle ACM$	Property S.S.S.
(b)	$\angle ABM \cong \angle ACM$	corresponding angles of congruent triangles.

26. 40 inches

27. 100 feet

28. (a) 36 feet

(b) $\overline{BC} \cong \overline{CD}$ (construction)

$\angle ACB \cong \angle ECD$ (vertical angles)

$\angle ABC \cong \angle EDC$ (all right angles are congruent)

$\therefore$ $\triangle ABC \cong \triangle EDC$, Property A.S.A.

29. $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$ given

$\overline{AB} \cong \overline{CB}$

$\overline{DA} \cong \overline{DC}$

$\overline{BD} \cong \overline{BD}$ common side

$\triangle ABD \cong \triangle CBD$ Property S.S.S.

$\angle ABD \cong \angle CBD$ corresponding angles in congruent triangles.

30. (a) $\triangle ADC \cong \triangle CBA$

$\triangle ADB \cong \triangle CBD$

$\triangle ADO \cong \triangle CBO$

$\triangle ABO \cong \triangle CDO$

(b) $\angle BAO \cong \angle DCO$

$\angle ABO \cong \angle CDO$

$\angle AOB \cong \angle COD$

(c) Yes, opposite sides of a parallelogram or corresponding sides of congruent triangles.

Chapter 5

RELATIVE ERROR

General Remarks

The suggested time allotment for this chapter is 6 days.

The general aim of this chapter is to make the pupils aware of the approximate character of many of the calculations which they carry out. Usually, this approximate aspect is introduced through some measurement or some experimental determination of a physical quantity.

It is important for them to realize that calculations with "exact" numbers, carried out correctly, still may yield "approximate" results when related to the physical quantities described in a problem. It is especially important for them to develop some appreciation of the circumstances when extreme accuracy in numerical work is warranted and when it is meaningless. They should be encouraged to remain alert for later problems which illustrate the main points of this chapter.

The chapter discussion is just a brief introduction to the subject. The pupils will benefit a great deal from occasional questions and reminders in connection with later problems involving physical measurements.

It is a fundamental principle of scientific discipline that no direct measurement can be achieved with complete and total accuracy. (Accuracy here is being used in the sense of being correct, not in the technical sense that is developed later in the chapter. The teacher here might note that "accurate" as a word with a definite meaning and "correct" in the usual sense might be quite different.) Many errors may enter into the act of measurement. Some of the sources of error are:

(a) Mistakes
(b) Instrumental errors
(c) External errors
(d) Personal errors

Preventable mistakes come about through ignorance, carelessness, or improper use of the measuring instrument. These mistakes can be corrected by means of a system of checks, by proper education and by careful inspection during the measuring process.

Instrumental errors are inherent in the measuring device. These errors are the results of imperfections in materials and in the manufacturing process. They are also the end products of economies in the system of manufacturing. To make an instrument more precise requires better materials, more time and superior manufacturing techniques, all of which cause higher costs.

External errors are environmental errors introduced by causes known and unknown. Wind currents may affect the careful weighing of an object. Temperature changes may cause corresponding changes in the dimensions both of objects measured and of measuring instruments. Some of these errors can be controlled to a degree by measuring in a controlled environment. Some of the errors of known origin can be corrected by applying correction factors, which, in effect, nullify the external errors. Some external errors are so complicated in origin that they defy analysis and complete correction.

Personal errors are due to human imperfections. Most measurements have been made by human beings in one way or another. No two persons react to a situation in exactly the same manner. Nevertheless, personal errors can be reduced by proper selection and training of personnel in the measuring operation. (This latter factor may explain to the teachers why industries spend great sums of money simply to teach people how to read measuring instruments. Pupils may then see the consequent importance of this in school work.)

Errors are also characterized as determinate and indeterminate. We have already discussed some of the consequences of these errors under types of errors above. Determinate errors are ones that occur in a known pattern throughout a series of measurements. Such errors can be analyzed, accounted for, and corrected. Instrumental errors and external errors caused by changes in temperature are of the determinate variety.

Indeterminate errors are the more insidious and more difficult to recognize and cope with. They have no *modus operandi* and their occurrence is haphazard and inconsistent. Human errors introduced by fatigue and anxiety are of the indeterminate variety. A sudden gust of wind during a measuring operation may introduce an indeterminate error.

The unit selected for a given measurement should be suitable for both the thing to be measured and the purpose for which the measurement is to be used. The unit is not necessarily a standard unit, but it may be a multiple of a standard unit or a subdividion of a standard unit. For example, the height of an airplane above the ground may be stated using 100 feet as a unit. The height of a person may be stated to the nearest half-inch--that is, using the half-inch as a unit. The result of measurement should be stated so as to indicate what unit was used, but this is not always done. In this chapter, $2\frac{3}{4}$ in. implies that the unit used was $\frac{1}{4}$ inch, and that the result is stated to the nearest fourth-inch. The result, "$2\frac{3}{4}$ in.," thus applies to any measurement which is within half of the unit on either side of the $2\frac{3}{4}$ inch mark--that is, more than $2\frac{5}{8}$ and less than $2\frac{7}{8}$ inches. We have used the term "greatest possible error" to refer to the amount by which the actual measurement may vary from the stated result. The use of the word "error" may cause some difficulty. It should be emphasized that the word is used here to mean that a measurement, such as $2\frac{3}{4}$ inches may represent any measurement within the range $(2\frac{3}{4} - \frac{1}{8})$ inches to $(2\frac{3}{4} + \frac{1}{8})$ inches. This meaning should be distinguished from the more familiar use of the term to mean mistakes in using the measuring instrument, mistakes in reading the scale, or mistakes resulting from use of a faulty instrument, such as a poorly marked ruler.

The two terms, "precision" and "accuracy," are sometimes confusing. In common usage, precision is the size of the unit of measurement used--the smaller the unit, the more precise the measurement. This means, of course, that the more precise the measurement, the smaller is the greatest possible error. Accuracy of a measurement is the ratio of the greatest possible error to the measurement--that is, accuracy refers to the relative error.

In order to develop clearly the concepts of unit of measurement and greatest possible error, it is suggested that the students be given considerable practice in measuring lengths of line segments to the nearest inch, nearest half-inch, nearest tenth of an inch, and so on. They should be asked to state the greatest possible error for measurements and to indicate within what range a stated measurement must fall.

Students sometimes have difficulty in seeing why measurements with the same significant digits in the same order have the same accuracy, regardless of the unit of measurement. A development along the following lines is sometimes helpful. Significant digits are defined as the digits in the numeral which show the number of units. The measurements below are analyzed to show the unit, the number of units, and the possible error.

Measurement	Unit	Number of Units	Greatest Possible Error
3,570 ft.	10 ft.	357	5 ft.
0.0357 ft.	0.0001 ft.	357	0.00005 ft.

Each of these measurements contains three significant figures. To determine the accuracy, we find the relative error, or percent of error. In the first case, we have the ratio $\frac{5}{3750}$. In the second case, the ratio is $\frac{0.00005}{0.0357}$, which equals $\frac{5}{3750}$.

Students also have difficulty in understanding why measurements with a larger number of significant digits have greater accuracy. An example similar to that on the previous page can be used.

Measurement	Unit	Number of Units	Greatest Possible Error
3.57 ft.	0.01 ft.	357	0.005 ft.
35.70 ft.	0.01 ft.	3570	0.005 ft.

Relative error of 3.57 ft. is $\frac{0.005}{3.57}$, or $\frac{5}{3570}$.

Relative error of 35.70 ft. is $\frac{0.005}{35.70}$, or $\frac{5}{35700}$.

The student should constantly be cautioned that the rules presented for computation with approximate data are "rough" and are not universally applicable.

Bakst in his Approximate Computation* makes this important point:

> "Generally speaking, the technique of Approximate Computation is not mechanical. The performance of numerical processes may be thought of as mechanical, but the arithmetic of Approximate Computation can be fully appreciated if and only if the interpretative processes are predominant. Only when a student is conscious of the nature of the data and can interpret the approximativeness and the meaning of the numerical results obtained by him, will he understand the importance of Approximate Computation as a fundamental part of applied mathematics."

---

*Twelfth Yearbook of the National Council of Teachers of Mathematics, Washington, D.C.: National Council of Teachers of Mathematics, 1937, p. 18.

For instance, when we carry a computed number like π or a trigonometric function to one more significant digit than the least precise of the approximate factors, we are attempting to minimize the error introduced by the numbers arising from calculation rather than measurement.

If it seems desirable, this unit might be increased in scope by reports on such topics as the history of standard units of measurement, various measuring devices used in industry and science, and the work of the United States Bureau of Standards.

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1. School Mathematics Study Group. Studies in Mathematics, Vol. V. CONCEPTS OF INFORMAL GEOMETRY, by R. D. Anderson. Stanford University: School Mathematics Study Group. Chapter 7, Accuracy and Precision.

2. Swain, R. L.; UNDERSTANDING ARITHMETIC. New York: Rinehart and Co., 1957. pp. 231-49.

3. The National Council of Teachers of Mathematics. The GROWTH OF MATHEMATICAL IDEAS, GRADES K-12 (Twenty-Fourth Yearbook). Washington, D.C.: The National Council of Teachers of Mathematics, 1959. pp. 182-228.

5-1. Greatest Possible Error

Answers to Class Exercises 5-1

1. $\frac{1}{8}$ in.
2. (a) .1 cm. or 1 mm.
   (b) (3.7 ± .05) cm.
   (c) .05 cm. or .5 mm.
3. $\frac{1}{200}$ cm. = .005 cm. or .05 millimeters
4. Greatest possible error = $\frac{1}{2}$ of unit of measurement.
5. Between (0.350 - .0005) in. and (0.350 + .0005) in. or between .3495 in. and 0.3505 in.

5-2. Precision and Significant Digits

Answers to Exercises 5-2

1. 3.20 inches
2. 4.0 inches or (4.0 ± 0.05) inches.
3. (a) 5.2 feet

   (b) 0.68 feet

   (c) Both have the same precision.
4. $12\frac{1}{2}$
5. (a) a. 100 feet c. 10 feet e. 0.0001 foot
   b. 1 foot d. 0.1 foot f. 0.1 foot

   (b) a. 50 feet c. 5 feet e. 0.00005 foot
   b. 0.5 foot d. 0.05 foot f. 0.05 foot
6. (a) e is the most precise.

   (b) a is the least precise.

   (c) Yes; d and f have the same precision.

7. (a) $420\underline{0}$
   (b) $23,\underline{0}00$
   (c) $48,0\underline{0}0,000$

8. (a) 2
   (b) 4
   (c) 1
   (d) 4
   (e) 4
   (f) 2
   (g) 4
   (h) 3

9. (a) 4
   (b) 4
   (c) 2
   (d) 3
   (e) 5
   (f) 2

5-3. Relative Error, Accuracy, and Percent of Error

Answers to Exercises 5-3

1. (a) 0.5 foot
   (b) 0.05 inch
   (c) 5 miles
   (d) 0.5 foot
   (e) 0.005 inches
   (f) 0.0005 foot
   (g) 500 miles
   (h) 5 miles

2. (a) 0.0096
   (b) 0.012
   (c) 0.0019
   (d) 0.0014
   (e) 0.00071
   (f) 0.083
   (g) 0.0093
   (h) 0.000093

3. (a) 0.05 foot; 0.54%
   (b) 0.0005 foot; 0.54%
   (c) 5 feet; 0.54%
   (d) 500 feet; 0.54%

4. The percents of error are the same for each measurement. In each case, the greatest possible error was the same fractional part of the measurement.

5. (a) 0.1 foot
(b) 0.001 foot
(c) 10 feet
(d) 1,000 miles
(e) 0.1 foot
(f) 0.001 inch

6. (a) 3
(b) 4
(c) 3
(d) 4

7. (a) 0.00096
(b) 0.000096
(c) 0.0014
(d) 0.00014

8. Yes. As the number of significant digits increases the relative error decreases. The larger the number of significant digits the greater the accuracy.

9. 7,812 inches has the greatest accuracy.
0.2 inch has the least accuracy.

10. (a) $(36\frac{1}{2} \pm \frac{1}{4})$ in., $(22.25 \pm .125)$ in., $(46\frac{2}{7} \pm \frac{1}{14})$ in.
$(32\frac{3}{8} \pm \frac{1}{16})$ in., $(27 \pm \frac{1}{32})$ in.

(b) 3 inches, 82.4 inches, 4.62 inches, 3.041 inches, 0.3762 inches.

*11. 6 feet $\pm \frac{1}{2}$ foot.
$3\frac{1}{2}$ inches $\pm \frac{1}{8}$ inch.
7.2 miles ± 0.05 mile.
3 yards 4 inches $\pm \frac{1}{4}$ inch.
3.2 inches ± 0.005 inch.

12. (a) 4
(b) 4
(c) 5
(d) 2
(e) 4
(f) 2
(g) 5
(h) 4
(i) 5
(j) 2
(k) 2
(l) 8

13. (a) 4.63×10^{8}
    (b) 3.270×10^{5}
    (c) 4.62×10^{-4}
    (d) 3.2004×10^{1}
    (e) 2×10^{0}
    (f) 4.00×10^{-5}
    (g) 3.68×10^{6}
    (h) 8.0×10^{-8}
    (i) 7.2×10^{10}

14. e, j, i, c, g, a, f, b, d, h.

15. BRAINBUSTER. $\frac{.00005}{3\frac{1}{2}} = \frac{5 \times 10^{-5}}{3.5} = \frac{1}{7 \times 10^{4}}$

$$\frac{5 \times 10^{6}}{(8.6)(6 \times 10^{12})} = \frac{10^{7}}{(2)(8.6)(6 \times 10^{12})} = \frac{1}{1.032 \times 10^{7}}$$

If two fractions have the same numerator, then the one with the larger denominator has the smaller value. Since, by definition, a measurement with a smaller relative error is the more accurate, the astronomer made the more accurate measurement.

5-4. Adding and Subtracting Measures

Answers to Exercises 5-4

1. (a) $\frac{3}{4}$ inch.
   (b) $\frac{7}{8}$ inch.
   (c) 0.055 inch.
   (d) 0.15 inch.
   (e) 0.0510 inch.
   (f) $\frac{7}{32}$ inch.

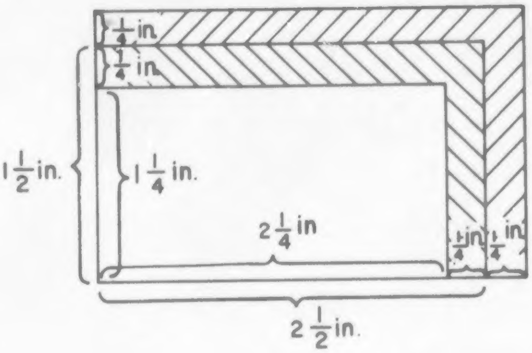
2. (a) 731.8
   (b) 145.1
   (c) 1,758.03

3. (a) 1.0
   (b) 734
   (c) 4780

5-5. Multiplying and Dividing Measures

Answers to Exercises 5-5

1. Largest area = $4\frac{13}{16}$ sq. in.
   Smallest area = $2\frac{13}{16}$ sq. in.
   Difference = 2 sq. in.
   Area is ($3\frac{3}{4}$ sq. in.) or 4 sq. in. (to the nearest $\frac{1}{2}$ sq. in.)



$\frac{1}{4}$ in.
$\frac{1}{4}$ in.
$1\frac{1}{2}$ in.
$1\frac{1}{4}$ in.
$2\frac{1}{4}$ in
$\frac{1}{4}$ in
$\frac{1}{4}$ in
$2\frac{1}{2}$ in.

2. (a) 150
   (b) 17,000
   (c) 11×10^4

3. (a) 4.4
   (b) 1.14
   (c) 3.92×10^5

4. 2.55×10^5 (or 255,000) square rods.

5. (a) 11 inches
   (b) 145.6 feet
   (c) 20 miles

6. 46.9 pounds. Three significant digits are used here because 75 is an exact counting number and does not affect the number of significant digits.

7. 250

Sample Questions

True-False

T 1. Counting separate objects is considered to be an exact process.

T 2. The smaller the unit, the more precise is the measurement.

F 3. The greatest possible error would be one-sixteenth inch if the length of a line is measured to the nearest $\frac{1}{4}$ inch.

T 4. The smaller the percent of error, the greater is the accuracy of the measurement.

F 5. If a measurement of a line is stated to be 10.0 inches, it implies that the line was measured to the nearest inch.

F 6. The more precise the measurement, the greater is the possible error.

T 7. A measurement of 300 miles has the same greatest possible error as a measurement of 700 miles.

T 8. There are four significant digits in the measurement 7,003 miles.

T 9. The term "greatest possible error" of a measurement does not refer to a mistake made in the measurement.

F 10. The greatest possible error of the sum of several approximate measurements is the same as the greatest possible error of the least precise measurement.

Multiple Choice

B 1. The most precise measurement is:

A. $(26\frac{1}{2} \pm \frac{1}{4})$ inches

B. 26.0 inches

C. 260 inches

D. $(26\frac{1}{4} \pm \frac{1}{8})$ inches

C 2. The number with the greatest accuracy is the one with the least:

A. precision

B. possible error

C. percent of error

D. number of significant digits

A 3. The greatest possible error in the sum of 45.5 in., 36.05 in., 5.1 in. is:

A. 0.105 in.

B. 0.05 in.

C. 0.055 in.

D. 0.005 in.

Completion

$\frac{1}{16}$ 4. The measurement of a line segment was stated to be $1\frac{1}{8}$ inches using $\frac{1}{8}$" as a unit.

The measurement might be stated as $1\frac{1}{8} \pm$ ______ inches.

$\frac{1}{2}$ 5. The greatest possible error in a measurement is always ________ of the unit used.

Other Sample Questions

6. (a) Compute the percent of error in the measurement 25.0 inches.

   (b) How would this compare with the percent of error in the measurement 25.0 miles?

ANSWER

(a) $\frac{.05}{25} = .002 = .2\%$

(b) Same

7. How many significant digits are there in each of the following numerals?

		ANSWER	
(a)	14.082	Five	(1, 4, 0, 8, 2)
(b)	9.600	Four	(9, 6, 0, 0)
(c)	0.0316	Three	(3, 1, 6)
(d)	19,414,500	Six	(1, 9, 4, 1, 4, 5)
(e)	16,$\underline{0}$00	Three	(1, 6, 0)
(f)	0.00024	Two	(2, 4)

8. Work out the greatest possible error in (86 + 18.48). Hint: Write 86 as 86 ± .5 and 18.48 as (18.48 ± .005).

ANSWER: (104.48 ± .505)

9. The dimensions of a room are measured as 16 ft. and 18 ft., correct to the nearest foot. What is the floor area and what are the largest and smallest possible areas corresponding to possible errors of measurement?

(ANSWER: Area, using measured dimensions, 288 sq. ft.
Greatest possible area 16.5 × 18.5 sq.ft. = 305.25 sq.ft.
Least possible area 15.5 × 17.5 sq.ft. = 271.25 sq.ft.)

Chapter 6

REAL NUMBERS

This chapter is intended to give a first brief and intuitive approach to the real number system. The time needed for the average class is estimated to be 15 days.

The spirit of the chapter is in keeping with the approach outlined in the appendices of the Report of the Commission on Mathematics of the College Entrance Examination Board, Chapter 1, Algebra, Section 3.

Principal emphases have been placed on three main points:

First, the irrational numbers do in fact exist, and are as "real" as the rational numbers with which the pupils now have some familiarity. The first introduction of an irrational number by geometrical construction of a point on the number line should strengthen this conviction, as should the approach via decimals.

Second, great stress is placed upon the one-to-one correspondence between the points of a line and the real numbers.

Third, the important property of density is treated in detail. It leads naturally to a development of the possibility of approximating an irrational number by rationals.

The development of decimal representations for rationals is intended to strengthen the pupil's grasp of the rational number system as well as to provide a plausible basis for the introduction of irrationals.

As a by-product of the treatment, the pupil should be expected to have gained a fair mastery of the properties of the rational number system and of the real number system. He should be expected to state the properties in his own words. The importance of density and closure particularly should be continually emphasized.

The treatments of the real number system in the following references will provide much worthwhile supplementary material.

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2. National Council of Teachers of Mathematics, Twenty-Third Yearbook. INSIGHTS INTO MODERN MATHEMATICS. Washington, D.C.: The Council, 1957. Chapter II, The Concept of Number.

3. National Council of Teachers of Mathematics, Twenty-Fourth Yearbook. THE GROWTH OF MATHEMATICAL IDEAS, Grades K-12; Washington, D.C.: The Council, 1959. Chapter 2, Number and Operation; Chapter 11, Promoting the Continuous Growth of Mathematical Concepts.

4. Courant, Richard and Robbins, Herbert. WHAT IS MATHEMATICS? New York: Oxford University Press, 1953. Chapters 1 and 2.

5. Richardson, M. FUNDAMENTALS OF MATHEMATICS. New York: Macmillan, 1941.

6. College Entrance Examination Board. REPORT OF THE COMMISSION ON MATHEMATICS, APPENDICES. Princeton, N.J.: College Entrance Examination Board, c/o Educational Testing Service, Box 592. Part 1, Algebra. See especially pp. 28-35, A Classroom Approach to Irrational Numbers.

7. Gamow, George. ONE, TWO, THREE ... INFINITY. New York: Viking Press, 1947. Chapter 1, Big Numbers. See especially pp. 14-23.

8. School Mathematics Study Group, Studies in Mathematics, Volume III. STRUCTURE OF ELEMENTARY ALGEBRA, by Vincent H. Haag. Stanford University: School Mathematics Study Group. Chapters 3, 4, and 5.

6-1. Review of Rational Numbers

Answers to Questions in Section 6-1

(a) _____, ______, $n - 1$, $n + 1$.

(b) Yes. One, because $n - 1$ would be zero and zero is not a counting number.

(c) Yes. No. The smallest is 1.

(d) (1) Yes (2) No (3) Yes (4) No.

(e) 2, 999, none, no.

Zero is neither positive or negative.

Answers to Class Exercise 6-1

1. No, Yes, $^{-}1$

2. $n + 1$, $n - 1$

3. (a) Yes
   (b) No
   (c) No
   (d) No

4. (a) $\frac{23}{4}$
   (b) $\frac{57}{8}$
   (c) $\frac{12}{1}$
   (d) $\frac{47}{100}$
   (e) $^{-}(\frac{5}{3})$
   (f) $\frac{69}{10}$
   (g) $\frac{37}{10}$
   (h) $^{-}(\frac{20}{3})$

5. (a) Identity for addition.
   (b) Closure under addition.
   (c) Commutative property of multiplication.
   (d) Distributive property of multiplication over addition.
   (e) Identity for multiplication.

(f) Associative property of addition.

(g) Closure under multiplication.

6. $\frac{7}{4}$ 7. $^-(\frac{4}{7})$

8. Reciprocal 9. 1

10. If they are the same when written in simplest form.
If the numerators are equal when the denominators are equal.

11. $\frac{10}{14}$, $\frac{15}{21}$, $\frac{20}{28}$, etc.

Answers to Exercises 6-1

1. (a) Closure under addition.
   (b) Identity for addition.
   (c) Identity for multiplication.
   (d) Closure under multiplication.
   (e) Distributive property of multiplication over addition.
   (f) Commutative property of multiplication.
   (g) Associative property of addition.

2. (a) $\frac{35}{2}$ (d) $^-(\frac{7}{20})$
   (b) $^-(\frac{5}{7})$ (e) $\frac{10}{1}$
   (c) $^-(\frac{11}{3})$ (f) $\frac{1703}{100}$

3. (a) $\frac{1}{2}$ (d) $\frac{44}{7}$
   (b) $\frac{3}{4}$ (e) $^{-}(\frac{33}{4})$
   (c) $^{-}(\frac{31}{50})$ (f) $\frac{25}{2}$

4. (a) $^{+}28$ or 28 (c) $^{-}(3\frac{1}{7})$ or $^{-}(\frac{22}{7})$
   (b) $^{-}756$ (d) $^{+}\frac{176}{5}$ or $\frac{176}{5}$

5. One 6. Zero

7. $^{-}4$, $^{-}(\frac{2}{3})$, 0, $\frac{3}{8}$, $\frac{2}{5}$, 0.41, $\frac{7}{16}$, $\frac{4}{7}$

8. $^{-}2$

9. No. The average of an odd integer and an even integer is never an integer.
   $\frac{1}{2}(^{-}3 + 8) = 2\frac{1}{2}$

10. (a) 3.3333 (d) 1.42142
    (b) 0.90909 (e) 134.6333
    (c) 163.1212 (f) 8464.646

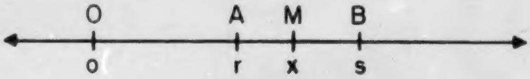
11. (a) 33.333 (d) 14.2142
    (b) 9.0909 (e) 1346.333
    (c) 1631.212 (f) 84646.46

12. (a) 333.33 (d) 142.142
    (b) 90.909 (e) 13463.33
    (c) 16312.12 (f) 846464.6

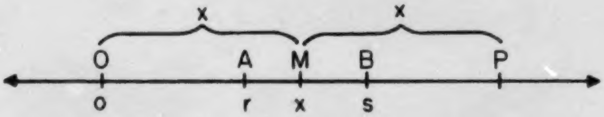
6-2. Density of Rational Numbers

The exercises on inserting more and more fractions in their proper order between 0 and 1 are intended to introduce the feeling of density without the precise statement of the property.

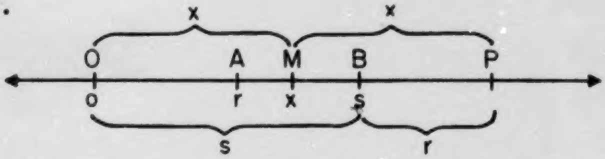
The development of the coordinate for the mid-point of a segment is incomplete. It was thought that a complete discussion might divert the attention of some of the pupils from the central idea of density. If you wish to fill in the gaps, you might go through the following argument with your class, perhaps using specific numbers for r and s. Consider two rational numbers r and s and the corresponding points A and B on the number line. Let M be the mid-point of the segment $\overline{AB}$.



Is M a rational point? If x is the number corresponding to M, is x rational? Let us investigate. Construct a point P to the right of M with MP = x = OM.



Since OM = MP and AM = MB, we must have OA = BP. But OA = r, since A is the point corresponding to the number r. Thus, BP = r.



From this we see two ways of finding OP. OP = x + x = 2x and also OP = r + s. Thus, 2x and r + s are names for the same number, so

$$2x = r + s.$$

If "twice x" is $r + s$ then x must be $\frac{1}{2}(r + s)$;

$$x = \frac{1}{2}(r + s).$$

Thus, M is a rational point corresponding to the average of r and s.

Your class may be interested in discovering other ways of finding rational numbers between rational numbers. They may think of methods by themselves. Here are two which you might suggest if you wish.

A. Suppose $\frac{a}{b} < \frac{c}{d}$. Write $\frac{a}{b} = \frac{ap}{bp}$, $\frac{c}{d} = \frac{cq}{dq}$ where $bp = dq$ and bp is positive. (That is, write fractions with positive common denominators for the two numbers.) It is then easy to find a rational number between $\frac{a}{b}$ and $\frac{c}{d}$: If $ap < n < cq$, then $\frac{ap}{bp} < \frac{n}{bp} < \frac{cq}{dq}$. However, if $cq = ap + 1$ there is no such integer n. In this case multiply $\frac{ap}{bp}$ and $\frac{cq}{dq}$ by $\frac{2}{2}$ and proceed as before. For example, to find a rational number between $\frac{10}{11}$ and $\frac{11}{12}$ we can first multiply $\frac{10}{11} \times \frac{12}{12}$ and $\frac{11}{12} \times \frac{11}{11}$ obtaining $\frac{120}{132}$ and $\frac{121}{132}$. Since there is no integer between 120 and 121 we now multiply both numbers by $\frac{2}{2}$. Thus, the two numbers will be $\frac{240}{264}$ and $\frac{242}{264}$. A number between them is now seen to be $\frac{241}{264}$.

B. Suppose $0 < \frac{a}{b} < \frac{c}{d}$, and a, b, c, and d are counting numbers. If we write $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$ we have $\frac{ad}{bd} < \frac{bc}{bd}$. Since the denominators are equal we must have $ad < bc$. Using this same method, compare $\frac{a}{b}$ and $\frac{a + c}{b + d}$. We write

$$\frac{a}{b} = \frac{a(b + d)}{b(b + d)} \text{ and}$$

$$\frac{a + c}{b + d} = \frac{b(a + c)}{b(b + d)}.$$

The denominators are the same. What about the numerators? $a(b + d) = ab + ad$ and $b(a + c) = ab + bc$, by the distributive property. Since $ad < bc$ we have $ab + ad < ab + bc$ so

$$\frac{a}{b} < \frac{a + c}{b + d}.$$

Similarly, we could show that

$$\frac{a + c}{b + d} < \frac{c}{d},$$

so $\frac{a + c}{b + d}$ is a rational number between $\frac{a}{b}$ and $\frac{c}{d}$. Caution: the pupils should not confuse $\frac{a + c}{b + d}$ with $\frac{a}{b} + \frac{c}{d}$.

For example, given that $0 < \frac{2}{3} < \frac{7}{8}$.

We write $\frac{2}{3} = \frac{2}{3} \cdot \frac{8}{8} = \frac{16}{24}$ and $\frac{7}{8} \cdot \frac{3}{3} = \frac{21}{24}$, and see that $16 < 21$.

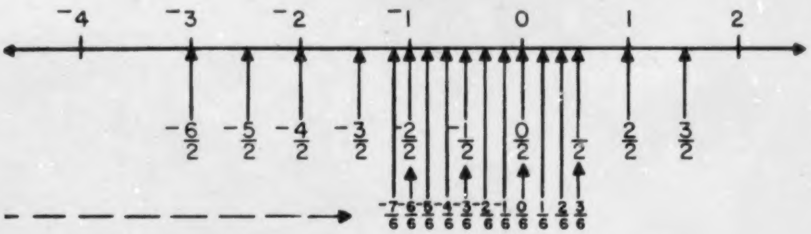
Now let us look at $\frac{2 + 7}{3 + 8}$ or $\frac{9}{11}$ and compare it with $\frac{2}{3}$.

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{11}{11} = \frac{22}{33} \quad \text{and} \quad \frac{9}{11} = \frac{9}{11} \cdot \frac{3}{3} = \frac{27}{33}.$$

So, $\frac{2}{3} < \frac{9}{11}$ and now we look at $\frac{9}{11}$ with respect to our other number $\frac{7}{8}$. We see that $\frac{9}{11} = \frac{9}{11} \cdot \frac{8}{8} = \frac{72}{88}$ and $\frac{7}{8} = \frac{7}{8} \cdot \frac{11}{11} = \frac{77}{88}$, so $\frac{9}{11} < \frac{7}{8}$ and therefore $\frac{2}{3} < \frac{9}{11} < \frac{7}{8}$. Another way then to find a rational between two positive rationals is to add the number in the numerator for the new numerator, and add the numbers in the denominator for the new denominator.

Answers to Exercises 6-2a

1.


-4
-3
-2
-1
0
1
2
$-\frac{6}{2}$
$-\frac{5}{2}$
$-\frac{4}{2}$
$-\frac{3}{2}$
$-\frac{2}{2}$
$-\frac{1}{2}$
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$
$\frac{3}{2}$
$-\frac{7}{6}$
$-\frac{6}{6}$
$-\frac{5}{6}$
$-\frac{4}{6}$
$-\frac{3}{6}$
$-\frac{2}{6}$
$-\frac{1}{6}$
$\frac{0}{6}$
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{3}{6}$

2.

3. Yes. $^-(\frac{6}{6})$ and $^-(\frac{2}{2})$, $^-(\frac{3}{6})$ and $^-(\frac{1}{2})$, $\frac{0}{6}$ and $\frac{0}{2}$, $\frac{3}{6}$ and $\frac{1}{2}$.

4. Six new points. $\frac{8}{7}$, $\frac{9}{7}$, $\frac{10}{7}$, $\frac{11}{7}$, $\frac{12}{7}$, $\frac{13}{7}$.

   Six new points. $\frac{22}{7}$, $\frac{23}{7}$, $\frac{24}{7}$, $\frac{25}{7}$, $\frac{26}{7}$, $\frac{27}{7}$.

5. Four new points. The first integer plus $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.

6. (a) $\frac{0}{3}$ names the same point as $\frac{0}{1}$.

   (b) $\frac{2}{4}$ names the same point as $\frac{1}{2}$.

   (c) In the row for sixths, $\frac{2}{6}$, $\frac{3}{6}$, and $\frac{4}{6}$ have already been named. Since 5 and 7 are prime numbers, points named by rational numbers with denominators 5 and 7 have not yet been named.

7. $\frac{1}{8} > 0$. $\frac{1}{8} = \frac{7}{56}$ and $\frac{1}{7} = \frac{8}{56}$, $\frac{8}{56} > \frac{7}{56}$.

   $\frac{1}{7} = \frac{6}{42}$ and $\frac{1}{6} = \frac{7}{42}$, $\frac{7}{42} > \frac{6}{42}$.

   $\frac{1}{6} = \frac{5}{30}$ and $\frac{1}{5} = \frac{6}{30}$, $\frac{6}{30} > \frac{5}{30}$.

   $\frac{1}{5} = \frac{4}{20}$ and $\frac{1}{4} = \frac{5}{20}$, $\frac{5}{20} > \frac{4}{20}$.

   $\frac{3}{4} = \frac{15}{20}$ and $\frac{4}{5} = \frac{16}{20}$, $\frac{16}{20} > \frac{15}{20}$.

   $\frac{4}{5} = \frac{24}{30}$ and $\frac{5}{6} = \frac{25}{30}$, $\frac{25}{30} > \frac{24}{30}$.

   $\frac{5}{6} = \frac{35}{42}$ and $\frac{6}{7} = \frac{36}{42}$, $\frac{36}{42} > \frac{35}{42}$.

   $\frac{6}{7} = \frac{48}{56}$ and $\frac{7}{8} = \frac{49}{56}$, $\frac{49}{56} > \frac{48}{56}$.

   $\frac{1}{1} = \frac{8}{8}$ and $\frac{8}{8} > \frac{7}{8}$.

8. Six. Four. Ten.

9. Eleven. Prime number. See answer to Problem 6(c).

Answers to Exercises 6-2b

1. No. There is no integer between $^{-}5$ and $^{-}4$.

2. Yes, 1 is the smallest. No largest positive integer.

3. No smallest negative integer. $^{-}1$ is the largest.

4. No. No.

5. $\frac{1}{200}$. $\frac{1}{400}$. $\frac{3}{400}$.

6. $\frac{3}{40}$, $\frac{5}{80}$ or $\frac{1}{16}$, $\frac{7}{80}$.

7. One such plan would be to use some point other than the halfway point. For example, the point $\frac{1}{3}$ of the way from $\frac{1}{1000}$ to $\frac{2}{1000}$ is $(\frac{1}{1000} + \frac{1}{3000})$ or $\frac{4}{3000}$. The next named points would be $(\frac{1}{1000} + \frac{1}{9000})$ or $\frac{10}{9000}$; $(\frac{1}{1000} + \frac{1}{27,000})$ or $\frac{28}{27,000}$; $(\frac{1}{1000} + \frac{1}{81,000})$ or $\frac{82}{81,000}$; $(\frac{1}{1000} + \frac{1}{243,000})$ or $\frac{244}{243,000}$.

6-3. Decimal Representations for Rationals

Note that we are considering a terminating decimal such as 3.75 as a periodic decimal $3.75\overline{0}\ldots$. Thus, we are not considering two kinds of decimals, terminating and periodic, but only periodic (infinite) decimals. This enables us to say, conveniently, that every rational number has a periodic decimal representation. Point this out to your class. You might say that we call such decimals as $3.750\overline{0}\ldots$ terminating not because they "end," but because the ordinary process of long division which we might use to find this decimal terminates:

$$\begin{array}{r} 3.75 \\ 4\,\overline{)15.00} \\ \underline{12} \\ 30 \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

We could carry this calculation no further. There is no need for it.

Answers to Exercises 6-3

1. (a) 2.2500000000...
   (b) 0.2083333333...
   (c) 0.4285714285...
   (d) 0.0857142857...
   (e) 0.02439024390...
   (f) 1.0240000000...
   (g) .378378378...
   (h) 0.0121012101...
   (i) 0.0121951219...
   (j) 0.058823529411764705...

2. a, c, d, g, i.

3. (a) 2
   (c) $2 \cdot 2$
   (d) 5
   (g) $2 \cdot 2 \cdot 2$
   (i) $2 \cdot 5$

4. (a) 0.142857
   (b) 0.285714
   (c) 0.428571
   (d) 0.571429
   (e) 0.714286
   (f) 0.857143

6-4. The Rational Number Corresponding to a Periodic Decimal

After reading this section and working the exercises, the pupil should understand that to find a fraction for the number n, represented by a repeating decimal he multiplies the number by 10^k, where k is the number of digits in the repeating group, and subtracts the original number from this product. The difference will always be a terminating decimal which will then be equal to $(10^k - 1)n$. After obtaining n as

$$\frac{\text{(terminating decimal)}}{10^k - 1}$$

it may be necessary to multiply numerator and denominator by a power of 10 in order to obtain a fraction for n of the form $\frac{p}{q}$, p and q integers. For example, to find the fraction for the decimal $n = 31.7257\overline{57}\ldots$ we multiply by 10^2 because there are two digits in the repeating group. Then we subtract n from $100n$,

$$\begin{array}{rl} 100n = & 3172.57\overline{57}\ldots \\ n = & 31.72\overline{57}\ldots \\ \hline 99n = & 3140.85 \end{array}$$

$$\text{and} \qquad n = \frac{3140.85}{99}.$$

To change this to the form $\frac{p}{q}$ where p and q are integers it is now necessary to multiply by $\frac{10^2}{10^2}$, obtaining $\frac{314085}{9900}$. In this case it is also possible to factor and "reduce,"

$$\frac{3 \cdot 5 \cdot 20939}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 11} = \frac{20939}{660}.$$

This is in simplest form since none of the prime factors of 660 (2, 3, 5, 11) are factors of 20939.

Answers to Class Exercise 6-4

1. (a) $9n$
   (b) $99n$
   (c) $990n$
   (d) $90n$
   (e) $999n$
   (f) $9{,}900n$

2. (a) $9.9\overline{9}\ldots$
   (b) $312.12\overline{12}\ldots$
   (c) $35.035\overline{035}\ldots$
   (d) $166.6\overline{6}\ldots$
   (e) $0.04\overline{4}\ldots$
   (f) $613.45\overline{345}\ldots$
   (g) $803.15\overline{15}\ldots$
   (h) $31289.9\overline{9}\ldots$
   (i) $3128.9\overline{9}\ldots$
   (j) $60123.0123\overline{0123}\ldots$

3. (a) $2816.10\overline{0}\ldots$ or 2816.1
   (b) $9.0\overline{0}\ldots$ or 9
   (c) $162.0\overline{0}\ldots$ or 162
   (d) $298.0\overline{0}\ldots$ or 298
   (e) $1.110\overline{0}\ldots$ or 1.11
   (f) $351.0\overline{0}\ldots$ or 351
   (g) $27048.0\overline{0}\ldots$ or 27048
   (h) $374.830\overline{0}\ldots$ or 374.83

4. (a)
$$\begin{array}{r} 10n = 5.5\overline{5}\ldots \\ n = 0.5\overline{5}\ldots \\ \hline 10n - n = 5.0\overline{0}\ldots \end{array}$$
   (b)
$$\begin{array}{r} 100n = 73.73\overline{73}\ldots \\ n = 0.73\overline{73}\ldots \\ \hline 100n - n = 73.0\overline{0}\ldots \end{array}$$
   (c)
$$\begin{array}{r} 1000n = 901.901\overline{901}\ldots \\ n = 0.901\overline{901}\ldots \\ \hline 1000n - n = 901.0\overline{0}\ldots \end{array}$$
   (d)
$$\begin{array}{r} 10n = 30.233\overline{3}\ldots \\ n = 3.023\overline{3}\ldots \\ \hline 10n - n = 27.210\overline{0}\ldots \end{array}$$

(e)
$$\begin{array}{r} 10n = 1{,}631.77\overline{7}\ldots \\ n = 163.17\overline{7}\ldots \\ \hline 10n - n = 1{,}468.60\overline{0}\ldots \end{array}$$

(f)
$$\begin{array}{r} 100n = 67{,}242.42\overline{42}\ldots \\ n = 672.42\overline{42}\ldots \\ \hline 100n - n = 66{,}570.0\overline{0}\ldots \end{array}$$

(g)
$$\begin{array}{r} 100n = 12.3456\overline{56}\ldots \\ n = 0.1234\overline{56}\ldots \\ \hline 100n - n = 12.22220\overline{0}\ldots \end{array}$$

(h)
$$\begin{array}{r} 10n = 34.100\overline{0}\ldots \\ n = 3.410\overline{0}\ldots \\ \hline 10n - n = 30.690\overline{0}\ldots \end{array}$$

5. (a) $\frac{31}{990}$ (d) $\frac{103}{99900}$

(b) $\frac{411}{900}$ (e) $\frac{3824}{90}$

(c) $\frac{163}{990}$ (f) $\frac{47531}{9999000}$

6. 25, 32, and 40.

Answers to Exercises 6-4

1. (a) $\frac{1}{11}$ (e) $\frac{1625}{10000} = \frac{65}{400} = \frac{13}{80}$

(b) $\frac{1}{9}$ (f) $\frac{1}{6}$

(c) $\frac{1}{18}$ (g) $\frac{5120}{999}$

(d) $\frac{41}{333}$ (h) $\frac{221}{22}$

2. (a) $32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ or 2^5

(b) $100 = 2 \cdot 2 \cdot 5 \cdot 5$ or $2^2 \cdot 5^2$

(c) $9 = 3 \cdot 3$ or 3^2

(d) $50 = 2 \cdot 5 \cdot 5$ or $2 \cdot 5^2$

(e) $35 = 5 \cdot 7$

(f) $80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $2^4 \cdot 5$

(g) $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ or $2^3 \cdot 3 \cdot 5$

(h) $160 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $2^5 \cdot 5$

3. The numbers in Parts a, b, d, f, h.

4. 64, 80, and 100.

6-5. Rational Points on the Number Line

The important understanding which the pupil should acquire in reading this section is that each repeating decimal corresponds to a unique point on the number line. We can locate the point with more and more accuracy by using finer and finer subdivisions on the number line and simultaneously looking at more and more of the digits in the decimal. Emphasize that there is only one point corresponding to the entire decimal and that by locating P with more and more accuracy, we are only getting closer and closer to that point.

Answers to Exercises 6-5

1. (a) 1.372 1.379 1.385 1.493 5.468

   (b) $^{-}9.426$ $^{-}5.630$ $^{-}2.765$ $^{-}2.763$ $^{-}2.761$

   (c) $^{-}0.15475$ 0.15463 0.15467 0.15475 0.15598

2. (a) none

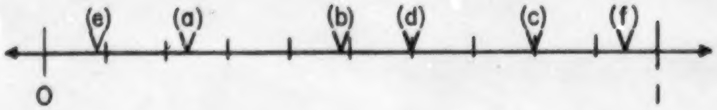
   (b) all but $^{-}0.15475$

   (c) all but $^{-}0.15475$

   (d) all but $^{-}0.15475$

   (e) 0.15475 0.15467 0.15463

3.



4. (a) $\frac{3}{9} = 0.3\overline{3}\ldots$, $\frac{17}{50} = 0.340\overline{0}\ldots$, $\frac{4}{10} = 0.40\overline{0}\ldots$

(b) $\frac{2}{3} = 0.6\overline{6}\ldots$, $\frac{67}{100} = 0.670\overline{0}\ldots$, $\frac{7}{10} = 0.70\overline{0}\ldots$

(c) $\frac{3}{7} = 0.\overline{428571}\ldots$, $\frac{4}{9} = 0.4\overline{4}\ldots$

(d) $\frac{152}{333} = 0.456\overline{456}\ldots$, $\frac{415}{909} = 0.4565\overline{4565}\ldots$

6-6. Irrational Numbers

This section contains one of the very few formal proofs in this volume. The whole purpose of the chapter may be defeated if the pupil is allowed to become mired in the details. The ideas involved in the proof are elementary, and an eighth grader should be able to follow the line of reasoning. However, it is far more important that the pupil be convinced by the proof of the irrationality of $\sqrt{2}$ than that he master the proof to the point of being able to reproduce it. Naturally, your attitude as a teacher will be an important factor in shaping the pupil's reaction. You should be thoroughly acquainted with the details of the proof and treat it as a natural line of discussion, rather than making a "big thing" of it.

If the statement, "if a^2 is even then a is even," troubles the pupil, the following explanation may be helpful:

> Remember the unique factorization property: A counting number may be factored as a product of primes in only one way (except for the order of the factors). The prime factors of a^2 are just the prime factors of a, each one taken twice in the product.

If 2 is one of the primes in the factorization of a^2 (that is, if a^2 is even) then it can be there only by being one of the prime factors of a (and there must then be two factors 2 in a^2). Thus, a must be even, too.

This same reasoning shows that if a prime p divides a^2 then p must also divide a. This is useful in proving that $\sqrt{p}$ is irrational if p is a prime. The case $p = 5$ is an exercise for the pupil (Problem 8, Exercises 6-6.).

You may wish to extend the discussion of the method of proof which we call "indirect reasoning." Many examples of indirect reasoning can be found in everyday life. What parent has never heard, "If you loved me, you would let me ... !" This is a fragment of an indirect proof of the "fact" that the parent does not love the child. (Fortunately, such "facts" usually dry up with the tears.) "If you loved me" is the assumption of the opposite of the statement the child obviously wishes to prove at the moment. The conclusion, "you would let me...," apparently contradicts something which has just become an established fact. The assumed statement "You love me," must, therefore, be very false.

A less facetious example is the proof of the statement:

> The number 0 has no inverse under multiplication. Suppose we assume that 0 has an inverse b. Then, by definition of a multiplicative inverse, $0 \cdot b = 1$. This is a direct contradiction of the facts that $0 \cdot x = 0$, for all numbers x, and $0 \neq 1$. Thus, our assumption that 0 has an inverse must be false.

The classical name for this type of reasoning is reductio ad absurdum. It is still common to refer to a "reductio proof."

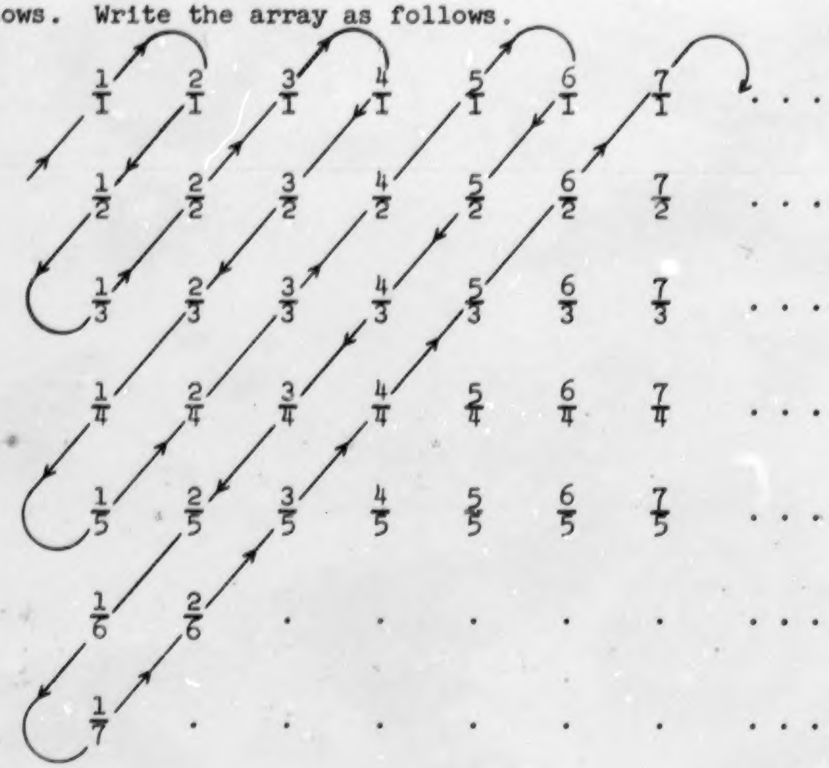
Another excellent example of a reductio proof is contained in the reference to Gamow's One, Two, Three, ... Infinity on the question of the non-enumerability of the set of irrational numbers.

[pages 257-260]

One actually proves that the set of real numbers between 0 and 1 cannot be enumerated, by assuming that they can be enumerated by some scheme and then constructing a decimal which has been missed by the enumeration. The assumption that you have enumerated the real numbers between 0 and 1 leads to the conclusion that you have not.

The fact that the set of real numbers is not enumerable, while the set of rational numbers is, implies that the irrational numbers themselves form a set which is not enumerable.

One of the really important distinctions between the rational number system and the system of irrationals is that you can show how to display all the rationals. One scheme is to proceed as follows. Write the array as follows.


1/1 2/1 3/1 4/1 5/1 6/1 7/1 . . .
1/2 2/2 3/2 4/2 5/2 6/2 7/2 . . .
1/3 2/3 3/3 4/3 5/3 6/3 7/3 . . .
1/4 2/4 3/4 4/4 5/4 6/4 7/4 . . .
1/5 2/5 3/5 4/5 5/5 6/5 7/5 . . .
1/6 2/6 . . . .
1/7 . . . .

We can find the positive rational $\frac{4}{5}$ in the 4th column and in the fifth row. In what row and what column would you look for the rational $\frac{9}{10}$? For $\frac{8}{17}$? For $\frac{p}{q}$?

By following the snaky line in the display on the previous page we can show a one-to-one correspondence between the set of positive rationals and the set of counting numbers like this:

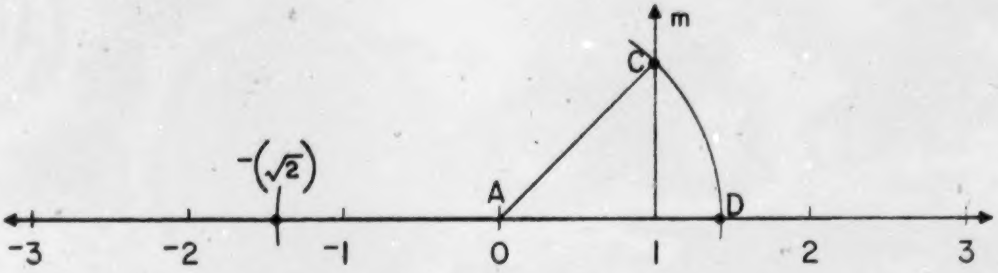
Counting numbers	1	2	3	4	5	6	7	8	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
Rational numbers	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	...

In this listing of the rational numbers we have followed the snaky line, but we have left out all fractions which are not in simplest form, because they are only other names for numbers already in our list. In this display $\frac{1}{2}$ is the 3rd rational number, $\frac{4}{1}$ is the 6th rational number; what is the 8th rational number? the 11th? If the above were continued, $\frac{5}{3}$ would be the what-th rational number?

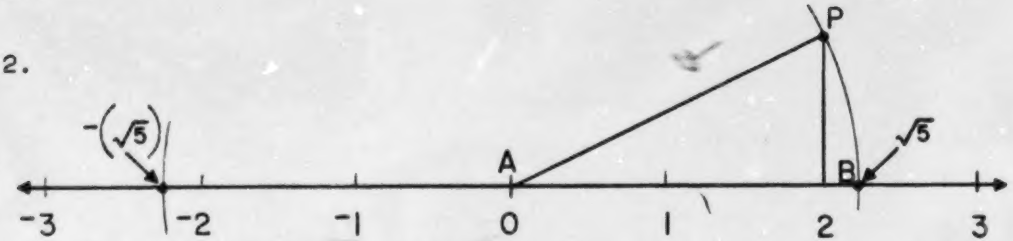
When we have set up a one-to-one correspondence between a given set and the set of counting numbers (or a subset of the set of counting numbers), mathematicians say we have "enumerated" the set. Thus, we have "enumerated" the set of positive rational numbers above.

Answers to Exercises 6-6

1.



2.

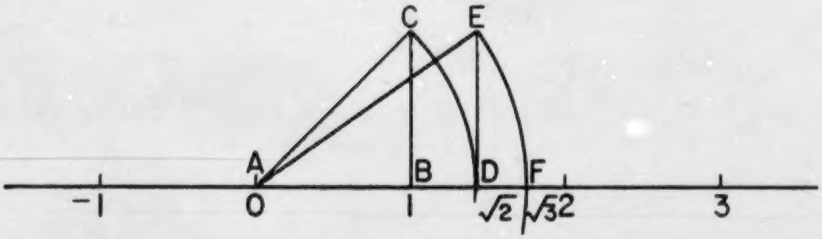


The measure of $\overline{AP}$ is $\sqrt{5}$.

3. See diagram for Problem 2.

4. Irrational. There is no rational number, $\frac{a}{b}$, such that $(\frac{a}{b})^2 = 5$.

*5.



$AB = 1$, $BC = 1$. So $AC = \sqrt{2}$. Make $AD = AC$, so point D represents $\sqrt{2}$.

Draw DE perpendicular to AD, $DE = 1$. Then $(AE)^2 = (\sqrt{2})^2 + 1^2$ and $AE = \sqrt{3}$. Make $AF = AE$, so point E corresponds to $\sqrt{3}$.

Similarly, $\sqrt{6}$ can be found by using a right triangle with sides of lengths $\sqrt{5}$ and 1; $\sqrt{7}$ by using a right triangle with sides of lengths $\sqrt{6}$ and 1.

6. Lay off the distance $\sqrt{2}$ twice to the right of 0; 3 times to the right of 0; three times to the left of 0.

7. Irrational.

*8. BRAINBUSTER. First show the following:

(a) If a number is divisible by 5, it can be expressed as $5 \cdot k$ where k is an integer.

(b) If a number is not divisible by 5, it can be expressed as $5n + R$, with $0 < R < 5$, R an integer. (Division Algorithm, Volume I, Chapter 5.)

(c) If n is divisible by 5, then n^2 is divisible by 5. [$(5k)^2 = 25k^2 = 5(5k^2)$. $5k^2$ is an integer by the closure property of integers under multiplication.]

(d) If n is not divisible by 5, then n^2 is not divisible by 5. If n is not divisible by 5, it may be expressed as: $5k + 1$, or $5k + 2$, or $5k + 3$, or $5k + 4$.

$(5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1$. $\therefore$ it is not divisible by 5.

$(5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$. $\therefore$ it is not divisible by 5.

$(5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$. $\therefore$ it is not divisible by 5.

$(5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1$. $\therefore$ it is not divisible by 5..

(e) If n is divisible by 5, and $\sqrt{n}$ is an integer, then $\sqrt{n}$ is divisible by 5.
Use an indirect argument.

Assume $\sqrt{n}$ is not divisible by 5. Then by (d) above, n is not divisible by 5. But n is divisible by 5. Therefore $\sqrt{n}$ is divisible by 5.
Now the proof follows the same form as the proof that $\sqrt{2}$ is not a rational number.

(1) Assume $\sqrt{5}$ is a rational number. Then $\sqrt{5} = \frac{p}{q}$, p and q integers and $q \neq 0$.

Let $\frac{p}{q}$ be the simplest expression for $\sqrt{5}$, so p and q have no common factors other than 1.

(2) $5 = \frac{p^2}{q^2}$.

(3) .·. $5q^2 = p^2$.

(4) q^2 is an integer.

(5) .·. p^2 is divisible by 5.

(6) .·. p is divisible by 5.

Let $p = 5k$, k an integer.

(7) $5q^2 = (5k)^2$ or $25k^2$.

(8) $q^2 = 5k^2$.

(9) k^2 is an integer.

(10) .·. q^2 is divisible by 5.

(11) .·. q is divisible by 5.

(12) By Statements 6 and 11, both p and q have the factor 5. But this contradicts the condition in Statement 1. Therefore the assumption that $\sqrt{5}$ is a rational number is false.

(13) .·. $\sqrt{5}$ is not a rational number.

6-7. A Decimal Representation for $\sqrt{2}$

Answers to Exercises 6-7

1. (a) $5 < \sqrt{30} < 6$ (d) $65 < \sqrt{4280} < 66$

   (b) $9 < \sqrt{89} < 10$ (e) $96 < \sqrt{9315} < 97$

   (c) $15 < \sqrt{253} < 16$

2. (a) 2.996361 (d) .003639 .000176 .003289

   (b) 2.999824 (e) 1.732

   (c) 3.003289

3. $(1.73)^2 = 2.9929$ $(1.74)^2 = 3.0276$ 1.73 is the better

4. $(3.87)^2 = 14.9769$ $(3.88)^2 = 15.0544$ 3.87 is the better

5. $(25.2)^2 = 635.04$ $(25.3)^2 = 640.09$ 25.2 is the better

6. 3.2 7. 12.2

8. 14.9 9. $n \approx 3.2$

10. $n \approx 12.2$

6-8. Irrational Numbers and the Real Number System

After students have read Sections 7 and 8, you might point out to them that, in obtaining a decimal representation for $\sqrt{2}$, we have not used any special properties of the number 2 other than the fact that it is a positive number and corresponds to a point on the number line. Any positive number N—rational or irrational—could be used in place of 2. Following the procedure of Section 7, we would obtain a sequence of decimals:

$$b_1b_2\ldots b_k.a_1,$$

$$b_1b_2\ldots b_k.a_1a_2,$$

$$b_1b_2\ldots b_k.a_1a_2a_3,$$

$$b_1b_2\ldots b_k.a_1a_2a_3a_4,$$

$$\ldots,$$

whose squares become closer and closer to N. The infinite decimal $b_1b_2\ldots b_k.a_1a_2a_3a_4\ldots$ which is obtained would represent a real number whose square is N. Thus, the real number system has this property: Every positive real number has a square root which is also a real number. A similar statement holds for arbitrary nth roots, where n is a counting number.

Answers to Exercises 6-8a

1.

	Rational		Irrational
(a)	$0.231\overline{231}\ldots$	(b)	$0.23123112311123\ldots$
(c)	$\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7}$	(d)	$\sqrt{7}$
(e)	$0.7\overline{8342}\ldots$	(f)	$\frac{\pi}{2}$
(i)	$0.750\overline{0}\ldots$	(g)	$\frac{3}{4}\sqrt{6}$
(j)	$\frac{58}{11}$	(h)	$9 - \sqrt{3}$
		(k)	$0.959559555955559\ldots$

Note that in (b) it is intended that the number of ones continues to increase.
Similarly, in (k) it is intended that the number of fives continues to increase.

2. (a) $0.231\overline{231}\ldots = \frac{231}{999} = \frac{77}{333}$

(c) $\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7} = 0.428571\overline{428571}\ldots$

(e) $0.78\overline{342}\ldots = \frac{78264}{99900} = \frac{2174}{2775}$

(i) $0.750\overline{0}\ldots = \frac{3}{4}$

(j) $\frac{58}{11} = 5.27\overline{27}\ldots$

3.

	Irrational Number	Nearest Hundredth
(b)	0.23123112311123...	0.23
(d)	$\sqrt{7}$	2.65
(f)	$\frac{\pi}{2}$	1.57
(g)	$\frac{3}{4}\sqrt{6}$	1.84
(h)	$9 - \sqrt{3}$	7.27
(k)	0.959559555955559...	0.96

Some possible answers for 4.

4. (a)	$.45632\overline{0}\ldots$	$8.27\overline{0}\ldots$	$.75\overline{0}\ldots$
(b)	$.45\overline{45}\ldots$	$12.762\overline{762}\ldots$	$.063\overline{63}\ldots$
(c)	.450450045...	.123112311123...	.565665666...

Answers to Exercises 6-8b

1. Possible answers are:

   (a) $2.375\overline{375}\ldots$ (b) $2.370370037000\ldots$

2. Possible answers are:

   (a) $.34\overline{4}\ldots$ (b) $.345634456344456\ldots$

3. Possible answers are:

   (a) $67.28\overline{2}\ldots$ (b) $67.28292020020002\ldots$

4. No. There are no square roots of negative integers in the real number system.

5. $\frac{355}{113} = 3.1415929\ldots$

   $\frac{22}{7} = 3.1428571\ldots$

   $\pi = 3.1415926\ldots$

   $\frac{355}{113}$ is a much better approximation to π than is $\frac{22}{7}$.

6-9. Geometric Properties of the Real Number System

Answers to Exercises 6-9

1. (a) irrational - circumference is π units.

   (b) rational - area is 1 square unit.

   (c) rational - hypotenuse is 13 units.

   (d) rational - $(\sqrt{3})^2 = 3$.

   (e) irrational - volume is 2π cubic units.

   (f) rational - (each side is $\sqrt{2}$ units).

2. $(1.414\ldots) \times (1.732\ldots) \approx 1.414 \times 1.732 \approx 2.449$

3. (a) $\sqrt{2} = (1.414\ldots) \qquad \frac{1}{2}\sqrt{2} = (0.707\ldots)$

$\sqrt{2} \times \frac{1}{2}\sqrt{2} = (1.414\ldots) \times (0.707\ldots)$

$\approx 1.414 \times 0.707 = 0.999698$

$\sqrt{3} + \sqrt{2} = (1.732\ldots) + (1.414\ldots) \approx 3.146$

$\sqrt{3} - \sqrt{2} = (1.732\ldots) - (1.414\ldots) \approx 0.318$

$(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} - \sqrt{2}) \approx (3.146) \times (0.318)$

$= 1.000428$

(b) $\sqrt{2} \times \frac{1}{2}\sqrt{2} = \frac{1}{2}(\sqrt{2})^2 = \frac{1}{2} \cdot 2 = 1$

$(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} - \sqrt{2}) = (\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} + {}^{-}\sqrt{2})$

$= (\sqrt{3} + \sqrt{2}) \cdot \sqrt{3} + (\sqrt{3} + \sqrt{2}) \cdot {}^{-}\sqrt{2}$

$= (\sqrt{3})^2 = (\sqrt{2} \cdot \sqrt{3}) + {}^{-}(\sqrt{3} \cdot \sqrt{2}) + {}^{-}(\sqrt{2})^2$

$= 3 + 0 + {}^{-}2$

$= 1$

4. Using 3.14 for π the radius $\approx \frac{1}{3.14}$

$\approx .318471\ldots$

Sample Questions

True-False

F 1. Every real number can be written as a rational number.

T 2. The smallest positive integer is one.

T 3. $\sqrt{2}$ is a number which when squared is equal to 2.

T 4. Three and one-seventh is a rational number.

T 5. Every repeating decimal is a rational number.

F 6. The square root of seven is 2.645.

T 7. Every real number can be represented by a point on the number line.

F 8. The number zero is not a rational number.

F 9. There are 12 integers between 15 and 27.

T 10. A rational number may be expressed as an integer divided by a counting number.

11. Fill in each blank with either true or false.

	The set has:	Counting Numbers	Integers	Rationals	Real Numbers
TTTT	closure under addition				
TTTT	closure under multiplication				
FTTT	closure under subtraction				
FFTT	closure under division except for division by zero				
FTTT	for each number an additive inverse.				
FFTT	for each number, except zero, a reciprocal.				
FFTT	has density.				

Multiple Choice

D 1. Which of the following is between $\frac{4}{9}$ and $\frac{2}{5}$?

A. $\frac{2}{3}$

B. $\frac{14}{16}$

C. $\frac{15}{16}$

D. $\frac{19}{45}$

E. $\frac{1}{3}$

C 2. Which of the following is equal to $\frac{5}{8}$?

A. 0.63

B. 0.750

C. 0.625

D. 0.562

E. 0.580

B 3. Which of the following is not a rational number?

A. $\frac{12}{5}$

B. $\frac{5}{0}$

C. $\frac{7}{2}$

D. 55

E. $\frac{27}{64}$

C 4. Which of the following is equal to $0.123\overline{123}\ldots$?

A. $\frac{1}{11}$

B. $\frac{1}{23}$

C. $\frac{41}{333}$

D. $\frac{3}{55}$

E. $\frac{123}{99}$

B 5. To change 670.313131 to a fraction, first multiply by:

A. 10

B. 100

C. 1000

D. 10,000

E. 100,000

Other Sample Questions

(Answers are at the end.)

6. Find two numbers between $\frac{10}{11}$ and $\frac{11}{12}$.

7. What rational number is represented by $0.63\overline{63}$?

8. What is the decimal name for $\frac{5}{80}$?

9. Which of these have decimal representations that repeat zero, $\frac{5}{6}$, $\frac{5}{8}$, $\frac{5}{9}$, $\frac{5}{15}$?

10. Write the closest integer to $\frac{35}{16}$ that is smaller than $\frac{35}{16}$.

11. Write an irrational number in decimal form.

12. Write a rational number between $0.625\overline{46}$ and $\frac{5}{8}$.

13. Which property is illustrated in the following sentence?
$\sqrt{5}\,(2 + 7) = (\sqrt{5} \cdot 2) + (\sqrt{5} \cdot 7)$

14. Draw a line segment that has a measure in inches of $\sqrt{6}$.

Answers to Other Sample Questions 6-14

6. Some examples are $\frac{241}{264}$, $\frac{361}{396}$, $\frac{362}{396}$, $\frac{481}{528}$.

7. $\frac{7}{11}$

8. 0.0625

9. $\frac{5}{8}$

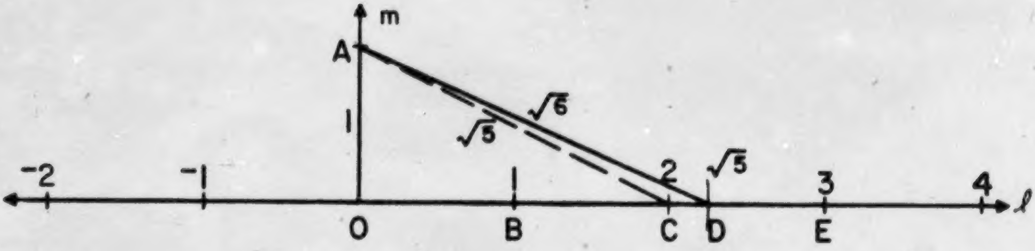
10. 2

11. One example is 3.747747774...

12. One example is $0.6253\overline{0}$

13. Distributive property.

14. Here is a method slightly different from the one in the text.


m
A
1
$\sqrt{6}$
$\sqrt{5}$
-2
-1
2
$\sqrt{5}$
3
4
ℓ
O
B
C
D
E

At zero construct line m perpendicular to ℓ.
With a compass measure $\overline{AC}$, whose measure is $\sqrt{5}$.
Mark off on line ℓ, with compass point at zero,
this distance of $\sqrt{5}$. Label this point D.
Draw $\overline{AD}$. This is the required segment whose measure
is $\sqrt{6}$.